INTRODUCTION

TO

QUATERNIONS,

WITH NUMEROUS EXAMPLES.

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QUATERNIONS,

WITH NUMEROUS EXAMPLES.

BY

P. KELLAND, M.A., F.R.S.,

FORMERLY FELLOW OF QUEENS' COLLEGE, CAMBRIDGE;

AND

P. G. TAIT, M.A.,

FORMERLY FELLOW OF ST PETER'S COLLEGE, CAMBRIDGE;

PROFESSORS IN THE DEPARTMENT OF MATHEMATICS IN THE UNIVERSITY OF EDINBURGH.

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PREFACE.

THE present Treatise is, as the title-page indicates, the joint production of Prof. Tait and myself. The preface I write in the first person, as this enables me to offer some personal explanations.

For many years past I have been accustomed, no doubt very imperfectly, to introduce to my class the subject of Quaternions as part of elementary Algebra, more with the view of establishing principles than of applying processes. Experience has taught me that to induce a student to think for himself there is nothing so effectual as to lay before him the different stages of the development of a science in something like the historical order. And justice alike to the student and the subject forbade that I should stop short at that point where, more simply and more effectually than at any other, the intimate connexion between principles and processes is made manifest. Moreover in lecturing on the groundwork on which the mathematical sciences are based, I could not but bring before my class the names of great men who spoke in other tongues and belonged to other nationalities than their own—Diophantus, Des Cartes, Lagrange, for instance—and it was not just to omit the name of one as

great as any of them, Sir William Rowan Hamilton, who spoke their own tongue and claimed their own nationality. It is true the name of Hamilton has not had the impress of time to stamp it with the seal of immortality. And it must be admitted that a cautious policy which forbids to wander from the beaten paths, and encourages converse with the past rather than interference with the present, is the true policy of a teacher. But in the case before us, quite irrespective of the nationality of the inventor, there is ample ground for introducing this subject of Quaternions into an elementary course of mathematics. It belongs to first principles and is their crowning and completion. It brings those principles face to face with operations, and thus not only satisfies the student of the mutual dependence of the two, but tends to carry him back to a clear apprehension of what he had probably failed to appreciate in the subordinate sciences.

Besides, there is no branch of mathematics in which results of such wide variety are deduced by one uniform process; there is no territory like this to be attacked and subjugated by a single weapon. And what is of the utmost importance in an educational point of view, the reader of this subject does not require to encumber his memory with a host of conclusions already arrived at in order to advance. Every problem is more or less self-contained. This is my apology for the present treatise.

The work is, as I have said, the joint production of Prof. Tait and myself. The preface I have written without consulting my colleague, as I am thus enabled

to say what could not otherwise have been said, that mathematicians owe a lasting debt of gratitude to Prof. Tait for the singleness of purpose and the self-denying zeal with which he has worked out the designs of his friend Sir Wm. Hamilton, preferring always the claims of the science and of its founder to the assertion of his own power and originality in its development. For my own part I must confess that my knowledge of Quaternions is due exclusively to him. The first work of Sir Wm. Hamilton, *Lectures on Quaternions*, was very dimly and imperfectly understood by me and I dare say by others, until Prof. Tait published his papers on the subject in the *Messenger of Mathematics.* Then, and not till then, did the science in all its simplicity develope itself to me. Subsequently Prof. Tait has published a work of great value and originality, *An Elementary Treatise on Quaternions.*

The literature of the subject is completed in all but what relates to its physical applications, when I mention in addition Hamilton's second great work, *Elements of Quaternions,* a posthumous work so far as publication is concerned, but one of which the sheets had been corrected by the author, and which bears all the impress of his genius. But it is far from elementary, whatever its title may seem to imply; nor is the work of Prof. Tait altogether free from difficulties. Hamilton and Tait write for mathematicians, and they do well, but the time has come when it behoves some one to write for those who desire to become mathematicians. Friends and pupils have urged me to undertake this duty, and after consultation with Prof. Tait, who from

being my pupil in youth is my teacher in riper years, I have, in conjunction with him, and drawing unreservedly from his writings, endeavoured in the first nine chapters of this treatise to illustrate and enforce the principles of this beautiful science. The last chapter, which may be regarded as an introduction to the application of Quaternions to the region beyond that of pure geometry, is due to Prof. Tait alone. Sir W. Hamilton, on nearly the last completed page of his last work, indicated Prof. Tait as eminently fitted to carry on happily and usefully the applications, mathematical and physical, of Quaternions, and as likely to become in the science one of the chief successors of its inventor. With how great justice, the reader of this chapter and of Prof. Tait's other writings on the subject will judge.

PHILIP KELLAND.

UNIVERSITY OF EDINBURGH,
October, 1873.

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INTRODUCTION TO QUATERNIONS.

CHAPTER I.

INTRODUCTORY.

THE science named Quaternions by its illustrious founder, Sir William Rowan Hamilton, is the last and the most beautiful example of extension by the removal of limitations.

The Algebraic sciences are based on ordinary arithmetic, starting at first with all its restrictions, but gradually freeing themselves from one and another, until the parent science scarce recognises itself in its offspring. A student will best get an idea of the thing by considering one case of extension within the science of Arithmetic itself. There are two distinct bases of operation in that science—addition and multiplication. In the infancy of the science the latter was a mere repetition of the former. Multiplication was, in fact, an abbreviated form of equal additions. It is in this form that it occurs in the earliest writer on arithmetic whose works have come down to us—Euclid. Within the limits to which his principles extended, the reasonings and conclusions of Euclid in his seventh and following Books are absolutely perfect. The demonstration of the rule for finding the greatest common measure of two numbers in Prop. 2, Book VII. is identically the same as that which is given in all modern treatises. But Euclid dares not venture on fractions. Their properties were probably all but unknown to him. Accordingly we look in vain for any *demonstration* of the properties of fractions in the writings of the Greek arithmeticians. For that we must come lower down. On the revival

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of science in the West, we are presented with categorical treatises on arithmetic. The first printed treatise is that of Lucas de Burgo in 1494. The author considers a fraction to be a quotient, and thus, as he expressly states, the order of operations becomes the reverse of that for whole numbers—multiplication precedes addition, etc. In our own country we have a tolerably early writer on arithmetic, Robert Record, who dedicated his work to King Edward the Sixth. The ingenious author exhibits his treatise in the form of a dialogue between master and scholar. The scholar battles long with this difficulty—that multiplying a thing should make it less. At first, the master attempts to explain the anomaly by reference to proportion, thus: that the product by a fraction bears the same proportion to the thing multiplied that the multiplying fraction does to unity. The scholar is not satisfied; and accordingly the master goes on to say: "If I multiply by more than one, the thing is increased; if I take it but once, it is not changed; and if I take it less than once, it cannot be so much as it was before. Then, seeing that a fraction is less than one, if I multiply by a fraction, it follows that I do take it less than once," etc. The scholar thereupon replies, "Sir, I do thank you much for this reason; and I trust that I do perceive the thing."

Need we add that the same difficulty which the scholar in the time of King Edward experienced, is experienced by every thinking boy of our own times; and the explanation afforded him is precisely the same admixture of multiplication, proportion, and division which suggested itself to old Robert Record. Every schoolboy feels that to multiply by a fraction is not to multiply at all in the sense in which multiplication was originally presented to him, viz. as an abbreviation of equal additions, or of repetitions of the thing multiplied. A totally new view of the process of multiplication has insensibly crept in by the advance from whole numbers to fractions. So new, so different is it, that we are satisfied Euclid in his logical and unbending march could never have attained to it. It is only by standing loose for a time to logical accuracy that extensions in the abstract sciences—extensions at any rate which stretch from one science to another—are effected. Thus Diophantus in his

Treatise on Arithmetic (i. e. Arithmetic extended to Algebra) boldly lays it down as a definition or first principle of his science that "minus into minus makes plus." The science he is founding is subject to this condition, and the results must be interpreted consistently with it. So far as this condition does not belong to ordinary arithmetic, so far the science extends beyond ordinary arithmetic: and this is the distance to which it extends—It makes subtraction to stand by itself, apart from addition; or, at any rate, not dependent on it.

We trust, then, it begins to be seen that sciences are extended by the removal of barriers, of limitations, of conditions, on which sometimes their very existence appears to depend. Fractional arithmetic was an impossibility so long as multiplication was regarded as abbreviated addition; the moment an extended idea was entertained, ever so illogically, that moment fractional arithmetic started into existence. Algebra, except as mere symbolized arithmetic, was an impossibility so long as the thought of subtraction was chained to the requirement of something adequate to subtract from. The moment Diophantus gave it a separate existence—boldly and logically as it happened—by exhibiting the law of *minus* in the forefront as the primary definition of his science, that moment algebra in its highest form became a possibility; and indeed the foundation-stone was no sooner laid than a goodly building arose on it.

The examples we have given, perhaps from their very simplicity, escape notice, but they are not less really examples of extension from science to science by the removal of a restriction. We have selected them in preference to the more familiar one of the extension of the meaning of an index, whereby it becomes a logarithm, because they prepare the way for a further extension in the same direction to which we are presently to advance. Observe, then, that in fractions and in the rule of signs, addition (or subtraction) is very slenderly connected with multiplication (or division). Arithmetic as Euclid left it stands on one support, addition only, inasmuch as with him multiplication is but abbreviated addition. Arithmetic in its extended form rests on two supports, addition and multiplica-

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tion, the one different from the other. This is the first idea we want our reader to get a firm hold of; that multiplication is not necessarily addition, but an operation self-contained, self-interpretable—springing originally out of addition; but, when full-grown, existing apart from its parent.

The second idea we want our reader to fix his mind on is this, that when a science has been extended into a new form, certain limitations, which appeared to be of the nature of essential truths in the old science, are found to be utterly untenable; that it is, in fact, by throwing these limitations aside that room is made for the growth of the new science. We have instanced Algebra as a growth out of Arithmetic by the removal of the restriction that subtraction shall require something to subtract from. The word "subtraction" may indeed be inappropriate, as the word multiplication appeared to be to Record's scholar, who failed to see how the multiplication of a thing could make it less. In the advance of the sciences the old terminology often becomes inappropriate; but if the mind can extract the right idea from the sound or sight of a word, it is the part of wisdom to retain it. And so all the old words have been retained in the science of Quaternions to which we are now to advance.

The fundamental idea on which the science is based is that of motion—of transference. Real motion is indeed not needed, any more than real superposition is needed in Euclid's Geometry. An appeal is made to mental transference in the one science, to mental superposition in the other.

We are then to consider how it is possible to frame a new science which shall spring out of Arithmetic, Algebra, and Geometry, and shall add to them the idea of motion—of transference. It must be confessed the project we entertain is not a project due to the nineteenth century. The Geometry of Des Cartes was based on something very much resembling the idea of motion, and so far the mere introduction of the idea of transference was not of much value. The real advance was due to the thought of severing multiplication from addition, so that the one might be the representative of a kind of motion absolutely different from that which was represented by

the other, yet capable of being combined with it. What the nineteenth century has done, then, is to divorce addition from multiplication in the new form in which the two are presented, and to cause the one, in this new character, to signify motion forwards and backwards, the other motion round and round.

We do not purpose to give a history of the science, and shall accordingly content ourselves with saying, that the notion of separating addition from multiplication—attributing to the one, motion from a point, to the other motion about a point—had been floating in the minds of mathematicians for half a century, without producing many results worth recording, when the subject fell into the hands of a giant, Sir William Rowan Hamilton, who early found that his road was obstructed—he knew not by what obstacle—so that many points which seemed within his reach were really inaccessible. He had done a considerable amount of good work, obstructed as he was, when, about the year 1843, he perceived clearly the obstruction to his progress in the shape of an old law which, prior to that time, had appeared like a law of common sense. The law in question is known as the *commutative* law of multiplication. Presented in its simplest form it is nothing more than this, "five times three is the same as three times five;" more generally, it appears under the form of "$ab = ba$ whatever a and b may represent." When it came distinctly into the mind of Hamilton that this law is not a necessity, with the extended signification of multiplication, he saw his way clear, and gave up the law. The barrier being removed, he entered on the new science as a warrior enters a besieged city through a practicable breach. The reader will find it easy to enter after him.

CHAPTER II.

VECTOR ADDITION AND SUBTRACTION.

1. *Definition of a Vector.* A vector is the representative of transference through a given distance, in a given direction. Thus if AB be a straight line, the idea to be attached to "vector AB" is that of transference from A to B.

For the sake of definiteness we shall frequently abbreviate the phrase "vector AB" by a Greek letter, retaining in the meantime (with one exception to be noted in the next chapter) the English letters to denote ordinary numerical quantities.

If we now start from B and advance to C in the same direction, BC being equal to AB, we may, as in ordinary geometry, designate "vector BC" by the same symbol, which we adopted to designate "vector AB."

Further, if we start from any other point O in space, and advance from that point by the distance OX equal to and in the same direction as AB, we are at liberty to designate "vector OX" by the same symbol as that which represents AB.

Other circumstances will determine the starting point, and individualize the line to which a specific vector corresponds. Our definition is therefore subject to the following condition :—*All lines which are equal and drawn in the same direction are represented by the same vector symbol.*

We have purposely employed the phrase "drawn in the same direction" instead of "parallel," because we wish to guard the student against confounding "vector AB" with "vector BA."

2. In order to apply algebra to geometry, it is necessary to impose on geometry the condition that when a line measured in one direction is represented by a *positive* symbol, the same line measured in the opposite direction must be represented by the corresponding *negative* symbol.

In the science before us the same condition is equally requisite, and indeed the reason for it is even more manifest. For if a transference from A to B be represented by $+\alpha$, the transference which neutralizes this, and brings us back again to A, cannot be conceived to be represented by anything but $-\alpha$, provided the symbols + and − are to retain any of their old algebraic meaning. The vector AB, then, being represented by $+\alpha$, the vector BA will be represented by $-\alpha$.

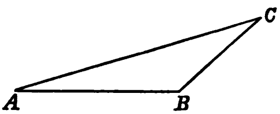
3. Further it is abundantly evident that so far as addition and subtraction of parallel vectors are concerned, *all* the laws of Algebra must be applicable. Thus (in Art. 1) $AB+BC$ or $\alpha+\alpha$ produces the same result at AC which is twice as great as AB, and is therefore properly represented by 2α; and so on for all the rest. The *distributive* law of addition may then be assumed to hold in all its integrity so long at least as we deal with vectors which are parallel to one another. In fact there is no reason whatever, so far, why α should not be treated in every respect as if it were an ordinary algebraic quantity. It need scarcely be added that vectors in the same direction have the same proportion as the *lines* which correspond to them.

We have then advanced to the following—

LEMMA. *All lines drawn in the same direction are, as vectors, to be represented by numerical multiples of one and the same symbol, to which the ordinary laws of Algebra, so far as their addition, subtraction, and numerical multiplication are concerned, may be unreservedly applied.*

4. The converse is of course true, that if lines as vectors are represented by multiples of the same vector symbol, they are parallel.

It is only necessary to add to what has preceded, that if BC be a line *not* in the same direction with AB, then the vector BC cannot be represented by α or by any multiple of α. The vector symbol α must be limited to express transference in a certain direction, and cannot, at the same time, express transference in any other direction. To express "vector BC" then, another and quite independent symbol β must be introduced. This symbol, being united to α by the signs + and −, the laws of algebra will, of course, apply to the combination.



5. If we now join AC, and thus form a triangle ABC, and if we denote vector AB by α, BC by β, AC by γ, it is clear that we shall be presented with the equation $\alpha + \beta = \gamma$.

This equation appears at first sight to be a violation of Euclid I. 20: "Any two sides of a triangle are together greater than the third side." But it is not really so. The anomalous appearance arises from the fact that whilst we have extended the meaning of the symbol + beyond its arithmetical signification, we have said nothing about that of a symbol =. It is clearly necessary that the signification of this symbol shall be extended along with that of the other. It must now be held to designate, as it does perpetually in algebra, "equivalent to." This being premised, the equation above is freed from its anomalous appearance, and is perfectly consistent with everything in ordinary geometry. Expressed in words it reads thus: "A transference from A to B followed by a transference from B to C is equivalent to a transference from A to C."

6. AXIOM. *If two vectors have not the same direction, it is impossible that the one can neutralize the other.*

This is quite obvious, for when a transference has been effected from A to B, it is impossible to conceive that any amount of transference whatever along BC can bring the moving point back to A.

It follows as a consequence of this axiom, that if α, β be *different* actual vectors, i.e. finite vectors not in the same direction, and if

$m\alpha + n\beta = 0$, where m and n are numerical quantities; then must $m = 0$ and $n = 0$.

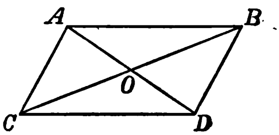
Another form of this consequence may be thus stated. If $m\alpha + n\beta = p\alpha + q\beta$, then must $m = p$, and $n = q$.

7. We now proceed to exemplify the principles so far as they have hitherto been laid down. It is scarcely necessary to remind the reader that we are assuming the applicability of all the rules of algebra and arithmetic, so far as we are yet in a position to draw on them; and consequently that our demonstrations of certain of Euclid's elementary propositions must be accepted subject to this assumption.

To avoid prolixity, we shall very frequently drop the word *vector*, at least in cases where, either from the introduction of a Greek letter as its representative, or from obvious considerations, it must be clear that the mere line is not meant. The reader will not fail to notice that the method of demonstration consists mainly in reaching the same point by two different routes. (See remark on Ex. 9.)

EXAMPLES.

Ex. 1. *The straight lines which join the extremities of equal and parallel straight lines towards the same parts are themselves equal and parallel.*



Let AB be equal and parallel to CD; to prove that AC is equal and parallel to BD.

Let vector AB be represented by α, then (Art. 1) vector CD is also represented by α.

If now vector CA be represented by β, vector DB by γ, we shall have (Art. 5)
$$\text{vector } CB = CA + AB = \beta + \alpha,$$
$$\text{and vector } CB = CD + DB = \alpha + \gamma;$$
$$\therefore \beta + \alpha = \alpha + \gamma,$$
$$\text{and } \beta = \gamma;$$
so that β and γ are the same vector symbol; consequently (Art. 1)

the *lines* which they represent are equal and parallel; i.e. CA is equal and parallel to BD.

Ex. 2. *The opposite sides of a parallelogram are equal; and the diagonals bisect each other.*

Since AB is parallel to CD, if vector AB be represented by α, vector CD will be represented by some numerical multiple of α (Art. 3), call it $m\alpha$.

And since CA is parallel to DB; if vector CA be β, then vector DB is $n\beta$; hence

$$\begin{aligned} \text{vector } CB &= CA + AB = \beta + \alpha, \\ \text{and } &= CD + DB = m\alpha + n\beta; \\ \therefore\ \alpha + \beta &= m\alpha + n\beta. \end{aligned}$$

Hence (Art. 6) $m = 1$, $n = 1$, i.e. the opposite sides of the parallelogram are equal.

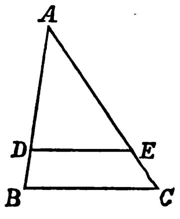
Again, as vectors, $AO + OB = AB$

$$\begin{aligned} &= CD \\ &= CO + OD; \end{aligned}$$

And as AO is a vector *along* OD, and CO a vector *along* OB; it follows (Art. 6) that vector AO *is* vector OD, and vector CO is OB;

$$\therefore \text{ line } AO = OD, \quad CO = OB.$$

Ex. 3. *The sides about the equal angles of equiangular triangles are proportionals.*



Let the triangles ABC, ADE have a common angle A, then, because the angles D and B are equal, DE is parallel to BC.

Let vector AD be represented by α, DE by β, then (Art. 3) AB is $m\alpha$, BC $n\beta$.

$$\begin{aligned} \therefore \text{ as vectors, } AE &= AD + DE = \alpha + \beta, \\ AC &= AB + BC = m\alpha + n\beta. \end{aligned}$$

Now AC is a multiple of AE, call it $p(\alpha + \beta)$.

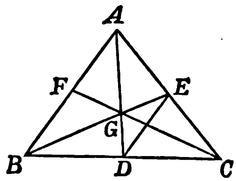
$$\begin{aligned} &\therefore\ m\alpha + n\beta = p(\alpha + \beta), \\ &\text{and } m = p = n \text{ (Art. 6)}. \end{aligned}$$

But line $AB : AD = m$,

line $BC : DE = n$,

$\therefore AB : AD :: BC : DE$.

Ex. 4. *The bisectors of the sides of a triangle meet in a point which trisects each of them.*



Let the sides of the triangle ABC be bisected in D, E, F; and let AD, BE meet in G.

Let vector BD or DC be α, CE or EA β, then, as vectors,

$$BA = BC + CA = 2\alpha + 2\beta = 2(\alpha + \beta),$$
$$DE = DC + CE = \alpha + \beta,$$

hence (Art. 4) BA is parallel to DE, and equal to $2DE$.

Again,
$$\begin{aligned} BG + GA &= BA \\ &= 2DE \\ &= 2(DG + GE). \end{aligned}$$

Now vector BG is along GE, and vector GA along DG.

$$\therefore \text{(Art. 6)} \quad BG = 2GE,$$
$$GA = 2GD,$$

whence the same is true of the lines.

Lastly,
$$\begin{aligned} BG &= \frac{2}{3} BE \\ &= \frac{2}{3}(BC + CE) \\ &= \frac{2}{3}(2\alpha + \beta); \end{aligned}$$

$$\begin{aligned} \therefore CG &= BG - BC \\ &= \frac{2}{3}(2\alpha + \beta) - 2\alpha \\ &= \frac{2}{3}(\beta - \alpha), \end{aligned}$$

$$GF = BF - BG$$
$$= \frac{1}{2} BA - BG,$$
$$= \alpha + \beta - \frac{2}{3}(2\alpha + \beta)$$
$$= \frac{1}{3}(\beta - \alpha);$$

hence CG is in the same straight line with GF, and equal to $2GF$.

Ex. 5. *When, instead of D and E being the middle points of the sides, they are any points whatever in those sides, it is required to find G and the point in which CG produced meets AB.*

Let $\frac{BC}{DC} = m$, $\frac{CA}{CE} = n$; also let vector $DC = \alpha$, vector $CE = \beta$;

$$\therefore\ BC = m\alpha,\ CA = n\beta.$$

Hence $\qquad BE = BC + CE = m\alpha + \beta,$

$$DA = \alpha + n\beta.$$

Let $\qquad BG = xBE,\ GA = yDA,$

then $\qquad BA = BG + GA = x(m\alpha + \beta) + y(\alpha + n\beta).$

But $\qquad BA = m\alpha + n\beta,$

$\therefore$ (Art. 6) $\quad xm + y = m,\ x + yn = n,$

and $\quad x$, i.e. $\frac{BG}{BE} = \frac{(m-1)n}{mn-1}$, y or $\frac{AG}{AD} = \frac{(n-1)m}{mn-1}$.

Again, let $\quad BF = pBA = p(m\alpha + n\beta).$

But $\qquad BF = BC + CF$

$$= m\alpha + \text{a multiple of } CG$$
$$= m\alpha + zCG \text{ suppose}$$
$$= m\alpha + z\{BG - BC\}$$
$$= m\alpha + z\left\{\frac{(m-1)n}{mn-1}(m\alpha + \beta) - m\alpha\right\}.$$

The two values of BF being equated, and Art. 6 applied, there results

$$p = 1 - z\frac{n-1}{mn-1},\quad p = z\frac{m-1}{mn-1},$$

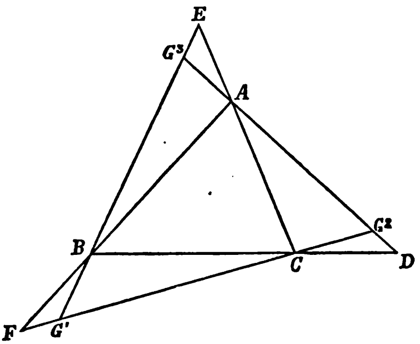
whence $$\frac{1-p}{p} = \frac{n-1}{m-1};$$

$$\text{i. e.} \quad \frac{AF}{BF} = \frac{AE}{CE} \div \frac{BD}{CD},$$

$$\text{or } AF.BD.CE = AE.CD.BF.$$

Ex. 6. *When, instead of as in* Ex. 4, *where D, E, F are points taken within BC, CA, AB at distances equal to half those lines respectively, they are points taken in BC, CA, AB produced, at the same distances respectively from C, A, and B; to find the intersections.*

Let the points of intersection be respectively G_1, G_2, G_3.



Retaining the notation of Ex. 4, we have

$$BD = 3\alpha, \quad CE = 3\beta;$$

and $\therefore$ $BG_3 = xBE$

$$= x(2\alpha + 3\beta) \quad \text{..........................(1)},$$

and $$BG_3 = BD + DG_3$$
$$= 3\alpha + yDA$$
$$= 3\alpha + y(CA - CD)$$
$$= 3\alpha + y(2\beta - \alpha);$$

$$\therefore\ 2x = 3 - y, \quad 3x = 2y, \quad \text{and } x = \frac{6}{7};$$

$$\therefore \text{ line } EG_3 = \frac{1}{7}EB.$$

$$\text{Similarly line } FG_1 = \frac{1}{7} FC,$$

$$\text{line } DG_2 = \frac{1}{7} DA,$$

and from equation (1) $BG_2 = \frac{6}{7}(2\alpha + 3\beta)$.

But $\quad BG_2 = BA + AG_2 = 2\alpha + 2\beta + AG_2;$

$$\therefore\ AG_2 = \frac{2}{7}(2\beta - \alpha);$$

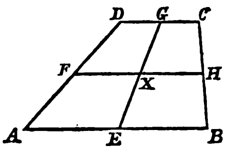
hence $\quad \text{line } AG_2 = \frac{2}{7} \text{ line } DA$

$$= 2DG_2,$$

and similarly of the others.

Ex. 7. *The middle points of the lines which join the points of bisection of the opposite sides of a quadrilateral coincide, whether the four sides of the quadrilateral be in the same plane or not.*

Let $ABCD$ be a quadrilateral; E, H, G, F the middle points of AB, BC, CD, DA; X the middle point of EG.



Let vector $AB = \alpha$, $AC = \beta$, $AD = \gamma$,

then $AE + EG = AD + DG$ gives

$$\frac{1}{2}\alpha + EG = \gamma + \frac{1}{2}(\beta - \gamma),$$

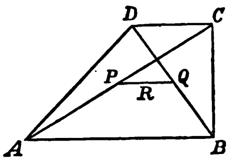
$$\text{and } AX = AE + \frac{1}{2}EG$$

$$= \frac{1}{4}(\alpha + \beta + \gamma),$$

which being symmetrical is α, β, γ in the same as the vector to the middle point of HF.

X is called (Art. 14) the mean point of $ABCD$.

Ex. 8. *The point of bisection of the line which joins the middle points of the diagonals of a quadrilateral (plane or not) is the mean point.*



Let P, Q be the middle points of AC, BD, R that of PQ.

Retaining the notation of the last example we have

$$AP = \frac{1}{2}\beta,$$

$$AQ = AB + BQ = \alpha + \frac{1}{2}(\gamma - \alpha) = \frac{1}{2}(\alpha + \gamma),$$

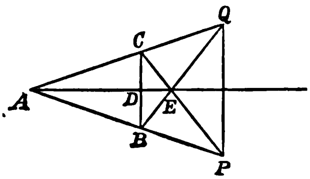
$$\text{i.e. } AQ = \frac{1}{2}(AB + AD).$$

Similarly $$AR = \frac{1}{2}(AP + AQ)$$

$$= \frac{1}{4}(\alpha + \beta + \gamma),$$

i.e. R is the same point as X in the last example; and is therefore the mean point of $ABCD$.

Ex. 9. *AD is drawn bisecting BC in D and is produced to any point E; AB, CE produced meet in P; AC, BE in Q; PQ is parallel to BC.*



Let $AB = \alpha$, $AC = \beta$,

$AP = x\alpha$, $AQ = y\beta$,

$\therefore\ BC = \beta - \alpha$, $AD = AB + \frac{1}{2}BC$,

$$= \frac{1}{2}(\alpha + \beta)$$

and AE is a multiple of $AD = z(\alpha + \beta)$ say.

Then $CP = pCE$ gives $x\alpha - \beta = p\{z(\alpha + \beta) - \beta\}$,

$$\therefore \text{(Art. 6) } x = pz,\ -1 = pz - p;$$

$$\therefore\ p = x + 1.$$

Similarly $BQ = qBE$ gives $y\beta - \alpha = q\{z(\alpha + \beta) - \alpha\}$,

$$y = qz,\ -1 = qz - 1,$$

$$\therefore\ q = y + 1,$$

and since $z = \frac{x}{p} = \frac{y}{q}$ we have

$$x = y, \; p = q;$$

$$\therefore \; PQ = y\beta - x\alpha = x(\beta - \alpha) = xBC,$$

hence the line PQ is parallel to BC.

The method pursued in this example leads to the solution of all similar problems. It consists, as we have already stated, in reaching the points P and Q respectively by two different routes,—viz. through C and through E for P; through B and through E for Q—and comparing the results.

Cor. 1. $PE : EC :: p - 1 : 1 :: x : 1 :: AP : AB.$

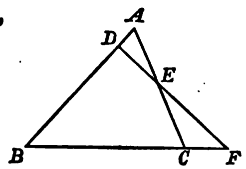
Cor. 2. $AE : AD :: 2z : 1 :: 2x : x + 1$

$$:: 2(p-1) : p$$

$$:: 2PE : PC,$$

$$\therefore \; AD : DE :: PE + EC : PE - EC.$$

Ex. 10. *If DEF be drawn cutting the sides of a triangle; then will $AD \,.\, BF \,.\, CE = AE \,.\, CF \,.\, BD$.*



Let $BD = \alpha$, $DA = p\alpha$, $AE = \beta$, $EC = q\beta$, then $BC = BA + AC = (1+p)\alpha + (1+q)\beta$, and CF is a multiple of BC.

Let $CF = xBC$

$$= x\{(1+p)\alpha + (1+q)\beta\}.$$

But
$$CF = CE + EF$$
$$= -EC + EF$$
$$= -q\beta + y(p\alpha + \beta);$$

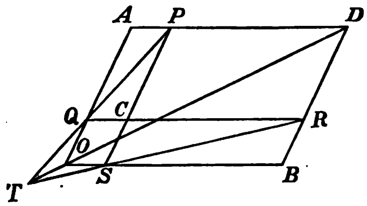
$\therefore$ equating, we have $x(1+p) = yp$, $x(1+q) = -q + y$,

whence
$$x = (1+x)pq,$$

i.e.
$$\frac{CF}{BC} = \frac{BF}{BC} \cdot \frac{AD}{BD} \cdot \frac{CE}{AE};$$

$$\therefore \; AD \,.\, BF \,.\, CE = AE \,.\, CF \,.\, BD.$$

Ex. 11. *If from any point within a parallelogram, parallels be drawn to the sides, the corresponding diagonals of the two*



parallelograms thus formed, and of the original parallelogram, shall meet in the same point.

Let PQ, RS meet in T;

join TO, OD.

Let $$OA = \alpha,\ OB = \beta,\ OQ = m\alpha,\ OS = n\beta,$$

then $QP = QC + CP = n\beta + (1-m)\alpha$, $SR = SC + CR = m\alpha + (1-n)\beta$,

and $$TO = TQ - OQ = x\{n\beta + (1-m)\alpha\} - m\alpha,$$

also $$TO = TS - OS = y\{m\alpha + (1-n)\beta\} - n\beta,$$

equating, there results

$$xn = y(1-n) - n;\ x(1-m) - m = ym;$$

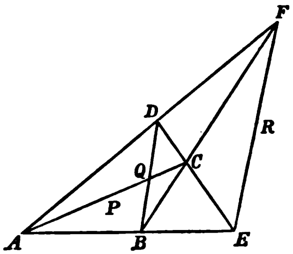
$$\therefore\ x = \frac{m}{1-m-n},$$

and $$TO = \frac{mn}{1-m-n}(\alpha+\beta) = \frac{mn}{1-m-n}OD;$$

hence (Art. 4) TO, OD are in the same straight line.

Cor. $TO : TD :: mn : (1-m)(1-n) :: OSCQ : CRDP.$

Ex. 12. *The points of bisection of the three diagonals of a complete quadrilateral are in a straight line.*



P, *Q*, *R* the middle points of the diagonals of the complete quadrilateral *ABCD*, are in a straight line.

Let $AB = \alpha$, $AD = \beta$,

$AE = m\alpha$, $AF = n\beta$;

$\therefore$ $BF = n\beta - \alpha$ and $BC = x(n\beta - \alpha)$,

$ED = \beta - m\alpha$ and $CD = y(\beta - m\alpha)$.

Now $BC + CD = BD = AD - AB$

gives $$x(n\beta - \alpha) + y(\beta - m\alpha) = \beta - \alpha,$$

whence $$xn + y = 1, \;\; x + my = 1,$$

$$\therefore \; x = \frac{m-1}{mn-1},$$

and $$AP = \frac{1}{2}AC = \frac{1}{2}\left\{\alpha + \frac{m-1}{mn-1}(n\beta - \alpha)\right\}$$

$$= \frac{1}{2}\frac{m(n-1)\alpha + n(m-1)\beta}{mn-1},$$

$$AQ = \frac{1}{2}(\alpha + \beta),$$

$$AR = \frac{1}{2}(m\alpha + n\beta),$$

$$\therefore \; AQ - AP = \frac{1}{2(mn-1)}\{(m-1)\alpha + (n-1)\beta\},$$

$$AR - AP = \frac{mn}{2(mn-1)}\{(m-1)\alpha + (n-1)\beta\},$$

or vector *PR* is a multiple of vector *PQ*, and therefore they are in the same straight line.

COR. Line $PQ : PR :: 1 : mn$

$:: AB \,.\, AD : AE \,.\, AF$

$::$ triangle ABD : triangle AEF.

We shall presently exemplify a very elegant method due to Sir W. Hamilton of proving three points to be in the same straight line.

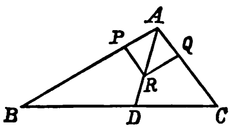
8. It is often convenient to take a vector of the length of the unit, and to express the vector under consideration as a numerical multiple of this unit. Of course it is not necessary that the unit should have any specified value; all that is required is that when once assumed for any given problem, it must remain unchanged throughout the discussion of that problem.

If the line AB be supposed to be a units in length, and the *unit vector* along AB be designated by α, then will *vector* AB be $a\alpha$ (Art. 3).

Sir William Hamilton has termed the length of the line in such cases, the TENSOR of the vector; so that the vector AB is the product of the tensor AB and the unit vector along AB. Thus if, as in the examples worked under the last article, we designate the vector AB by α, we may write $\alpha = T\alpha U\alpha$, where $T\alpha$ is an abbreviation for 'Tensor of the vector α'; $U\alpha$ for 'unit vector along α'.

EXAMPLES.

Ex. 1. *If the vertical angle of a triangle be bisected by a straight line which also cuts the base, the segments at the base shall have the same ratio that the other sides of the triangle have to one another.*



Take unit vectors along AB, AC, which call α, β respectively: construct a rhombus $APQR$ on them and draw its diagonal AR. Then since the diagonals of a rhombus bisect its angles, it is clear that the vector AD which bisects the angle A is a multiple of AR the diagonal vector of the rhombus.

Now $$AR = AP + PR = AP + AQ = \alpha + \beta,$$
$$\therefore AD = x(\alpha + \beta).$$

Now vector $AB = c\alpha$, $AC = b\beta$; using c, b as in ordinary geometry for the lengths of AB, AC.

Hence $$BD = AD - AB = x(\alpha + \beta) - c\alpha,$$
and $$BD = yBC = y(AC - AB)$$
$$= y(b\beta - c\alpha),$$

Equating, $x - c = -yc,\ x = yb$;

$$\therefore\ y = \frac{c}{b+c},$$

and
$$\begin{aligned} BD : DC &:: y : 1-y \\ &:: c : b \\ &:: BA : AC. \end{aligned}$$

COR. If α, β are unit vectors from A, and if δ be another vector from A such that $\delta = x(\alpha+\beta)$; then δ bisects the angle between α and β.

Ex. 2. *The three bisectors of the angles of a triangle meet in a point.*

Let AD, BE bisect A, B and meet in G, CG bisects C.

Let units along AB, AC, BC be α, β, γ, then as in the last example,

$$AG = x(\alpha+\beta),\ BG = y(-\alpha+\gamma).$$

But
$$a\gamma = b\beta - c\alpha,$$

$$\therefore\ BG = y\left(-\alpha + \frac{b\beta - c\alpha}{a}\right),$$

and
$$\begin{aligned} CG &= AG - AC \\ &= x(\alpha+\beta) - b\beta, \end{aligned}$$

also
$$\begin{aligned} CG &= BG - BC, \\ &= y\left(-\alpha + \frac{b\beta - c\alpha}{a}\right) - b\beta + c\alpha; \end{aligned}$$

$$\therefore\ x = -y - \frac{c}{a}y + c,$$

$$x - b = \frac{yb}{a} - b,$$

whence
$$x = \frac{bc}{a+b+c},$$

and
$$\begin{aligned} CG &= \frac{b}{a+b+c}\{c\alpha - (a+b)\beta\} \\ &= \frac{b}{a+b+c}(-a\gamma - a\beta) \\ &= p(\gamma+\beta), \end{aligned}$$

hence CG bisects the angle C (Cor. Ex. 1).

9. If α, β, γ are non-parallel vectors in the same plane, it is always possible to find numerical values of a, b, c so that $a\alpha + b\beta + c\gamma$ shall $= 0$.

For a triangle can be constructed whose sides shall be parallel respectively to α, β, γ.

Now if the vectors corresponding to those sides taken in order be $a\alpha, b\beta, c\gamma$ respectively, we shall have, by going round the triangle,

$$a\alpha + b\beta + c\gamma = 0.$$

10. If α, β, γ are three vectors neither parallel nor in the same plane, it is impossible to find numerical values of a, b, c, not equal to zero, which shall render $a\alpha + b\beta + c\gamma = 0$.

For (Art. 5) $a\alpha + b\beta$ can be represented by a third vector in the plane which contains two lines parallel respectively to α, β. Now $c\gamma$ is not in that plane, therefore (Art. 6) their sum cannot equal 0.

It follows that if $a\alpha + b\beta + c\gamma = 0$ and α, β, γ are not parallel vectors, they are in the same plane.

11. There is but one way of making the sum of multiples of α, β, γ (as in Art. 9) equal to 0.

Let $$a\alpha + b\beta + c\gamma = 0,$$
and also $$p\alpha + q\beta + r\gamma = 0.$$

By eliminating γ we get

$$(ar - cp)\,\alpha + (br - cq)\,\beta = 0\,;$$

$$\therefore \text{(Art. 6)}\ \ ar = cp,\ \ br = cq,$$

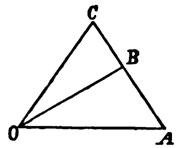
$$\text{or}\ \ a : b : c :: p : q : r,$$

so that the second equation is simply a multiple of the first.

12. If α, β, γ are coinitial, coplanar vectors terminating in a straight line, then the same values of a, b, c which render $a\alpha + b\beta + c\gamma = 0$ will also render $a + b + c = 0$.

Let vector $OA = \alpha$, $OB = \beta$, $OC = \gamma$, ABC being a straight line; then

$$AB = \beta - \alpha,$$
$$AC = \gamma - \alpha.$$



But AC is a multiple of AB,

$$\text{or } \gamma - \alpha = p(\beta - \alpha),$$
$$\text{i.e. } (p-1)\alpha - p\beta + \gamma = 0.$$

But $$(p-1) - p + 1 = 0;$$

and as $p-1$, $-p$, $+1$ correspond to a, b, c and satisfy the condition required, the proposition is proved generally (Art. 11).

13. Conversely, if α, β, γ are coinitial coplanar vectors, and if both $a\alpha + b\beta + c\gamma = 0$ and $a + b + c = 0$, then do α, β, γ terminate in a straight line.

For $$a\gamma + b\gamma + c\gamma = 0;$$

therefore by subtraction

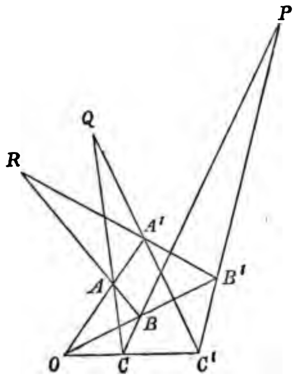
$$a(\gamma - \alpha) + b(\gamma - \beta) = 0,$$

i.e. $\gamma - \alpha$ is a multiple of $\gamma - \beta$, and therefore (Art. 4) in the same straight line with it: i.e. AC is in the same straight line with BC. (See Tait's *Quaternions*, § 30.)

EXAMPLES.

Ex. 1. *If two triangles are so situated that the lines which join corresponding angles meet in a point, then pairs of corresponding sides being produced will meet in a straight line.*

ABC, $A'B'C'$ are the triangles; O the point in which $A'A$, $B'B$, $C'C$ meet; P, Q, R the points in which BC, $B'C'$, &c. meet: PQR is a straight line.



Let $OA = \alpha$, $OB = \beta$, $OC = \gamma$,
$OA' = m\alpha$, $OB' = n\beta$, $OC' = p\gamma$,

then $$BA = \alpha - \beta,$$
and $$BR = x(\alpha - \beta);$$
$$B'A' = m\alpha - n\beta,$$
and $$B'R = y(m\alpha - n\beta).$$

Now $BB' = BR - B'R$ gives

$$(n-1)\beta = x(\alpha - \beta) - y(m\alpha - n\beta);$$

$$\therefore\ n - 1 = -x + ny,\ 0 = x - my,$$

and
$$x = -\frac{m(n-1)}{m-n};$$

whence
$$OR = OB + BR = \beta - \frac{m(n-1)}{m-n}(\alpha - \beta)$$
$$= \frac{n(m-1)\beta - m(n-1)\alpha}{m-n}.$$

Similarly,
$$OP = \frac{p(n-1)\gamma - n(p-1)\beta}{n-p},$$
$$OQ = \frac{m(p-1)\alpha - p(m-1)\gamma}{p-m};$$

$$\therefore\ (m-n)(p-1)\,OR + (n-p)(m-1)\,OP + (p-m)(n-1)\,OQ = 0.$$

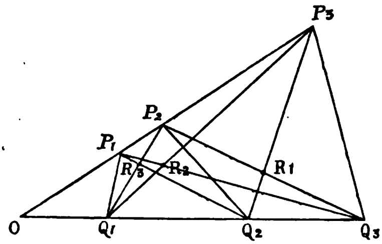
And also

$$(m-n)(p-1) + (n-p)(m-1) + (p-m)(n-1) = 0,$$

whence (Art. 13) P, Q, R are in the same straight line.

Ex. 2. *If a quadrilateral be divided into two quadrilaterals by any cutting line, the centres of the three shall lie in a straight line.*

Let $P_1Q_1Q_3P_3$ be the quadrilateral divided into two by the



line P_2Q_2. Let the diagonals of $P_2Q_2Q_3P_3$ meet in R_1; and so of the others: R_1, R_2, R_3 are the centres.

Produce P_3P_1, Q_3Q_1 to meet in O. Let unit vectors along OP, OQ be denoted by α, β; and put

$$OP_1 = m_1\alpha, \quad OP_2 = m_2\alpha, \quad OP_3 = m_3\alpha;$$
$$OQ_1 = n_1\beta, \quad OQ_2 = n_2\beta, \quad OQ_3 = n_3\beta;$$

then $$OR_3 = OP_1 + P_1R_3 = m_1\alpha + x(n_2\beta - m_1\alpha),$$

and $$OR_3 = OQ_1 + Q_1R_3 = n_1\beta + y(m_2\alpha - n_1\beta).$$

Equating, we have

$$m_1 - m_1x = m_2y, \text{ and } n_2x = n_1 - n_1y;$$
$$\therefore\ x = \frac{(m_1 - m_2)\,n_1}{m_1n_1 - m_2n_2},$$

and $$OR_3 = \frac{m_1m_2(n_1 - n_2)\,\alpha + n_1n_2(m_1 - m_2)\,\beta}{m_1n_1 - m_2n_2}.$$

Similarly,

$$OR_1 = \frac{m_2m_3(n_2 - n_3)\,\alpha + n_2n_3(m_2 - m_3)\,\beta}{m_2n_2 - m_3n_3},$$

$$OR_2 = \frac{m_1m_3(n_3 - n_1)\,\alpha + n_1n_3(m_3 - m_1)\,\beta}{m_3n_3 - m_1n_1};$$

$$\therefore\ (m_1n_1 - m_2n_2)\,m_3n_3\,OR_3 + (m_2n_2 - m_3n_3)\,m_1n_1\,OR_1$$
$$+ (m_3n_3 - m_1n_1)\,m_2n_2\,OR_2 = 0.$$

And also

$$(m_1n_1 - m_2n_2)\,m_3n_3 + (m_2n_2 - m_3n_3)\,m_1n_1$$
$$+ (m_3n_3 - m_1n_1)\,m_2n_2 = 0,$$

whence (Art. 13) R_1, R_2, R_3 are in the same straight line.

Cor. R_1, R_2, R_3 will pass through O provided the coefficients of α and β in the three vectors have the same proportion, i. e. provided

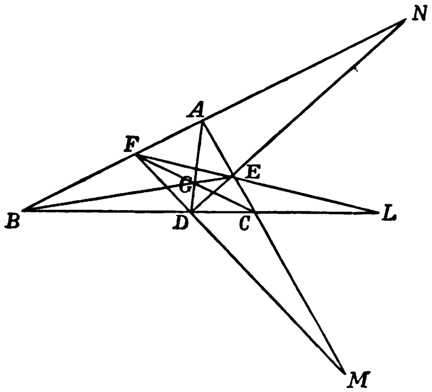
$$\frac{1}{m_1} - \frac{1}{m_2} : \frac{1}{m_2} - \frac{1}{m_3} :: \frac{1}{n_1} - \frac{1}{n_2} : \frac{1}{n_2} - \frac{1}{n_3}.$$

Ex. 3. *If AD, BE, CF be drawn cutting one another at any point G within a triangle, then FD, DE, EF shall meet the third sides of the triangle produced in points which lie in a straight line.*

Also the produced sides of the triangle shall be cut harmonically.

If, as in Ex. 5, Art. 7, we put

$$DC = \alpha,\ CE = \beta,\ BC = m\alpha,\ CA = n\beta,$$



we get, as in that example,

$$AF : BF :: n-1 : m-1;$$

$$\therefore\ BF = \frac{m-1}{m+n-2}(m\alpha + n\beta),$$

and $$FD = BD - BF = \frac{m-1}{m+n-2}\{(n-2)\,\alpha - n\beta\}.$$

$DM = xFD$, compared with

$$DM = DC - MC = \alpha - y\beta,$$

gives $$x\frac{(m-1)(n-2)}{m+n-2} = 1,\quad x\frac{(m-1)\,n}{m+n-2} = y;$$

$$\therefore\ y = \frac{n}{n-2},$$

and $$BM = BC - MC = m\alpha - \frac{n}{n-2}\beta.$$

Again, $$FE = FA + AE = \frac{n-1}{m+n-2}\{m\alpha - (m-2)\,\beta\}.$$

And $EL = xFE$, compared with

$$EL = CL - CE = y\alpha - \beta,$$

gives $$y = \frac{m}{m-2},$$

$$BL = (y+m)\,\alpha = \frac{m\,(m-1)}{m-2}\,\alpha.$$

Thirdly, $DN = xDE = x\,(\alpha+\beta)$, compared with

$$DN = BN - BD = y\,(m\alpha + n\beta) - (m-1)\,\alpha,$$

gives $$y = \frac{m-1}{m-n},$$

and $$BN = \frac{m-1}{m-n}(m\alpha + n\beta).$$

Now $$(m-1)(n-2)\,BM + (m-n)\,BN - (m-2)(n-1)\,BL = 0.$$

Also $(m-1)(n-2) + (m-n) - (m-2)(n-1) = 0$;

therefore BM, BN, BL are in a straight line (Art. 13).

Further, $$CL = \frac{m}{m-2}\,CD,$$

$$BL = \frac{m}{m-2}\,BD;$$

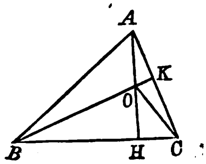
$$\therefore\ CL : CD :: BL : BD,$$

and BL is cut harmonically.

Ex. 4. *The point of intersection of bisectors of the sides of a triangle from the opposite angles, the point of intersection of perpendiculars on the sides from the opposite angles, and the point of intersection of perpendiculars on the sides from their middle points lie in a straight line which is trisected by the first of these points.*

1°. Let unit vector $CB = \alpha$, unit vector $CA = \beta$,

then Ex. 4, Art. 7, $$CG = \frac{1}{3}(a\alpha + b\beta).$$



2°. Let AH, BK perpendiculars on the sides intersect in O,

then
$$HA = b\beta - b\alpha \cos C,$$
$$= b(\beta - \alpha \cos C),$$
$$KB = a(\alpha - \beta \cos C).$$

Now $CO = CA + AO$, and also $= CB + BO$ gives

$$b\beta + yb(\beta - a\alpha \cos C) = a\alpha + xa(\alpha - \beta \cos C),$$

$$\therefore\ ax = \frac{b \cos C - a}{\sin^2 C},$$

and
$$CO = \frac{\cos C}{\sin^2 C}\{(b - a \cos C)\alpha + (a - b \cos C)\beta\}.$$

3°. Let perpendiculars from D and E (Ex. 4, Art. 7) meet in X,

then DX is a multiple of HA.

$$\therefore\ CX = CD + DX = CE + EX \text{ gives}$$

$$\frac{1}{2}a\alpha + v(\beta - \alpha \cos C) = \frac{1}{2}b\beta + z(\alpha - \beta \cos C),$$

$$\therefore\ v = \frac{b - a \cos C}{2 \sin^2 C},$$

and
$$CX = \frac{(a - b \cos C)\alpha + (b - a \cos C)\beta}{2 \sin^2 C},$$

$$\therefore\ 2CX + CO - 3CG = 0,$$

and also
$$2 + 1 - 3 = 0,$$

$\therefore$ X, O, G are in a straight line.

Also
$$CO - CG = 2(CG - CX),$$

or vector $GO = 2$ vector XG,

$$\therefore\ GO = 2GX,$$

and G trisects XO.

14. The vector to the mean point of any polygon is the mean of the vectors to the angles of the polygon.

1°. Let O be any point; then in the figure of Ex. 4, Art. 7 we have, calling OA, α, OB, β and OC, γ,

$$OG = \alpha + AG = \beta + BG = \gamma + CG$$

$$= \frac{1}{3}(\alpha + \beta + \gamma) + \frac{1}{3}(AG + BG + CG)$$

$$= \frac{1}{3}(\alpha + \beta + \gamma);$$

because $$AG + BG + CG = \frac{2}{3}(AD + BE + CF)$$

$$= \frac{2}{3}\{(AB + AC) + (BA + BC) + (CA + CB)\}$$

$$= 0.$$

2°. If OA, OB, OC, OD be α, β, γ, δ, in the figure of Ex. 7, Art. 7, we have

$$OX = OH + HX = OH + \frac{1}{2}(OF - OH)$$

$$= \frac{1}{2}(OF + OH) = \frac{1}{4}(\alpha + \beta + \gamma + \delta).$$

3°. In the more general case we may define the mean point in a manner analogous to that adopted in mechanics to define the centre of inertia of equal masses placed at the angular points of the figure. Thus, if we take any rectangular axes OX, OY, and designate by α, β unit vectors parallel to these axes; and by ρ_1, ρ_2, &c. the vectors to the different points; and if we write x_1, y_1; x_2, y_2, &c. for the Cartesian co-ordinates of the different points referred to those axes; and define the mean point as the centre of inertia of equal masses placed at the angular points; the Cartesian co-ordinates of that point will be

$$x = \frac{x_1 + x_2 + \ldots}{m}, \quad y = \frac{y_1 + y_2 + \ldots}{m},$$

and its vector $$\rho = x\alpha + y\beta.$$

Now $$\rho_1 = x_1\alpha + y_1\beta,\ \rho_2 = x_2\alpha + y_2\beta, \text{ \&c.}$$

$$\therefore\ \frac{\rho_1 + \rho_2 + \dots}{m} = \frac{x_1 + x_2 + \dots}{m}\alpha + \frac{y_1 + y_2 + \dots}{m}\beta$$

$$= x\alpha + y\beta,$$

$$= \rho.$$

COR. 1. $$(\rho_1 - \rho) + (\rho_2 - \rho) + (\rho_3 - \rho) + \text{\&c} = 0,$$

i. e. the sum of the vectors of all the points, drawn from the mean point, = 0.

The extension of the same theorem to three dimensions is obvious.

COR. 2. If we have another system of n points whose vectors are σ_1, σ_2, &c. then the vector to the mean point is

$$\sigma = \frac{\sigma_1 + \sigma_2 + \dots}{n}.$$

If now τ be the mean point of the whole system, we have

$$\tau = \frac{\rho_1 + \rho_2 + \dots + \sigma_1 + \sigma_2 + \dots}{m + n},$$

or $$(m + n)\,\tau - m\rho - n\sigma = 0,$$

hence (13) τ, ρ, σ terminate in a right line; or the general mean point is situated on the right line which connects the two partial mean points.

ADDITIONAL EXAMPLES TO CHAP. II.

1. If P, Q, R, S be points taken in the sides AB, BC, CD, DA of a parallelogram, so that $AP : AB :: BQ : BC$, &c., $PQRS$ will form a parallelogram.

2. If the points be taken so that $AP = CR$, $BQ = DS$, the same is true.

3. The mean point of $PQRS$ is in both cases the same as that of $ABCD$.

4. If $P'Q'R'S'$ be another parallelogram described as in Ex. 1, the intersections of PQ, $P'Q'$, &c. shall be in the angular points of a parallelogram $EFGH$ constructed from $PQRS$ as $P'Q'R'S'$ is constructed from $ABCD$.

5. The quadrilateral formed by bisecting the sides of a quadrilateral and joining the successive points of bisection is a parallelogram, with the same mean point.

6. If the same be true of any other equable division such as trisection, the original quadrilateral is a parallelogram.

7. If any line pass through the mean point of a number of points, the sum of the perpendiculars on this line from the different points, measured in the same direction, is zero.

8. From a point E in the common base AB of the two triangles ABC, ABD, straight lines are drawn parallel to AC, AD, meeting BC, BD at F, G; shew that FG is parallel to CD.

9. From any point in the base of a triangle, straight lines are drawn parallel to the sides: shew that the intersections of the diagonals of every parallelogram so formed lie in a straight line.

10. If the sides of a triangle be produced, the bisectors of the external angles meet the opposite sides in three points which lie in a straight line.

11. If straight lines bisect the interior and exterior angles at A of the triangle ABC in D and E respectively; prove that BD, BC, BE form an harmonical progression.

12. The diagonals of a parallelepiped bisect one another.

13. The mean point of a tetrahedron is the mean point of the tetrahedron formed by joining the mean points of the triangular faces; and also those of the edges.

14. If the figure of Ex. 11, Art. 7 be that of a gauche quadrilateral (a term employed by Chasles to signify that the triangles AOD, BOD are not in the same plane), the lines QP, DO, RS will

meet in a point, provided

$$\frac{AP}{PD} = m\frac{OS}{SB}, \text{and } \frac{AQ}{QO} = m\frac{DR}{RB}.$$

15. If through any point within the triangle ABC, three straight lines MN, PQ, RS be drawn respectively parallel to the sides AB, AC, BC; then will

$$\frac{MN}{AB} + \frac{PQ}{AC} + \frac{RS}{BC} = 2.$$

16. $ABCD$ is a parallelogram; E, the point of bisection of AB; prove that AC, DE being joined will trisect each other.

17. $ABCD$ is a parallelogram; PQ any line parallel to DC; PD, QC meet in S, PA, QB in R; prove that AD is parallel to RS.

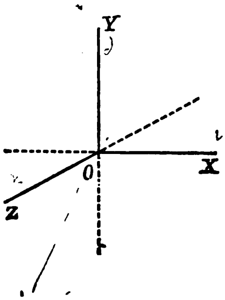
CHAPTER III.

VECTOR MULTIPLICATION AND DIVISION.

15. We trust we have made the reader understand by what we stated in our Introductory Chapter, that, whilst we retain for 'multiplication' all its old properties, so far as it relates to ordinary algebraical quantities, we are at liberty to attach to it any signification we please when we speak of the multiplication of a vector by or into another vector. Of course the interpretation of our results will depend on the definition, and may in some points differ from the interpretation of the results of multiplication of numerical quantities.

It is necessary to start with one limitation. Whereas in Algebra we are accustomed to use at random the phrases 'multiply by' and 'multiply into' as tantamount to the same thing, it is now impossible to do so. We must select one to the exclusion of the other. The phrase selected is 'multiply into'; thus we shall understand that the first written symbol in a sequence is the operator on that which follows: in other words that $\alpha\beta$ shall read 'α into β', and denote α operating on β.

16. As in the Cartesian Geometry, so here we indicate the position of a point in space by its relation to three axes, mutually at right angles, which we designate the axes of x, y, and z respectively. For graphic representation the axes of x and y are drawn in the plane of the paper whilst that of z being perpendicular to that plane is drawn in perspective only. As in ordinary



geometry we assume that when vectors measured forwards are represented by positive symbols, vectors measured backwards will be represented by the corresponding negative symbols. In the figure before us, the positive directions are *forwards*, *upwards* and *outwards;* the corresponding negative directions, *backwards*, *downwards* and *inwards*.

With respect to vector rotation we assume that, looked at in perspective in the figure before us, it is negative when in the direction of the motion of the hands of a watch, positive when in the contrary direction. In other words, we assume, as is done in modern works on Dynamics, that rotation is positive when it takes place from y to z, z to x, x to y: negative when it takes place in the contrary directions (see *Tait*, Art. 65).

Unit vectors at right angles to each other.

17. DEFINITION. If i, j, k be unit vectors along Ox, Oy, Oz respectively, the result of the multiplication of i into j or ij is defined to be the turning of j through a right angle in the plane perpendicular to i and in the positive direction; in other words, the operation of i on j turns it round so as to make it coincide with k; and therefore briefly $ij=k$.

To be consistent it is requisite to admit that if i instead of operating on j had operated on any other unit vector perpendicular to i in the plane of yz, it would have turned it through a right angle in the same direction, so that ik can be nothing else than $-j$. Extending to other unit vectors the definition which we have illustrated by referring to i, it is evident that j operating on k must bring it round to i, or $jk=i$.

Again, always remembering that the positive directions of rotation are y to z, z to x, x to y, we must have $ki=j$.

18. As we have stated, we retain in connection with this definition the old laws of numerical multiplication, whenever numerical quantities are mixed up with vector operations; thus $2i\,.\,3j=6ij$. Further, there can be no reason whatever, but the contrary, why the laws of addition and subtraction should undergo

any modification when the operations are subject to this new definition; we must clearly have

$$i\,(j+k) = ij + ik.$$

Finally, as we are to regard the operations of this new definition as operations of multiplication—magnitude and motion of rotation being united in one vector symbol as multiplier, just as magnitude and motion of translation were united in one vector symbol in the last chapter—we are bound to retain all the laws of algebraic multiplication so far as they do not give results inconsistent with each other. In no other way can the conclusions be made to compare with those deduced from the corresponding operations in the previous science. Thus we retain what Sir William Hamilton terms the *associative law of multiplication:* the law which assumes that it is indifferent in what way operations are grouped, provided the order be not changed; the law which makes it indifferent whether we consider abc to be $a \times bc$ or $ab \times c$. This law is *assumed* to be applicable to multiplication in its new aspect (for example that $ijk = ij \,.\, k$), and being assumed it limits the science to certain boundaries, and, along with other assumed laws, furnishes the key to the interpretation of results.

The law is by no means a necessary law. Some new forms of the science may possibly modify it hereafter. In the meantime the assumption of the law fixes the limits of the science.

The *commutative* law of multiplication under which order may be deranged, which is assumed as the groundwork of common algebra (we say *assumed* advisedly) is now no longer tenable. And this being the case it is found that the science of Quaternions breaks down one of the barriers imposed by this law and expands itself into a new field.

ij is *not* equal to ji, it is clearly impossible it should be.

A simple inspection of the figure, and a moment's consideration of the definition, will make this plain. The definition imposes on i as an operator on j the duty of turning j through a right angle as if by a left-handed turn with a cork-screw handle, thus throwing j *up* from the plane xy; when, on the other hand, j is the operator

and i the vector operated on, a similar left-handed turn will bring i *down* from the plane of xy. In fact $ij = k$, $ji = -k$, and so $ij = -ji$.

19. We go on to obtain one or two results of the application of the associative law.

1. Since $ij = k$, we have $i \,.\, ij = ik = -j$.

Now by the law in question,

$$i \,.\, ij = ii \,.\, j = i^2 \,.\, j\,;$$

$$\therefore\ i^2 \,.\, j = -j,$$

or $$i^2 = -1.$$

Our first result is that the square of the unit vector along Ox is -1; and as Ox may have any direction whatever, we have, generally, *the square of a unit vector* $= -1$. In other words, the repetition of the operation of turning through a right angle reverses a vector.

2. Again, $\quad ijk = i \,.\, jk = i \,.\, i = i^2 = -1.$

Similarly it may be proved that

$$jki = kij = -1,$$

or no change is produced in the product so long as direct cyclical order is maintained.

3. But $\quad ikj = i \,.\, kj = i \,.\, -i = -i^2 = +1\,;$

$$\therefore\ ijk = -ikj,$$

or a derangement of cyclical order changes the sign of the product. This last conclusion is also manifest from Art. 18.

Vectors generally not at right angles to each other.

20. We have already (Art. 8) laid down the principle of separation of the vector into the product of tensor and unit vector; and we apply this to multiplication by the considerations given in Art. 18, from which it follows at once that if α be a vector along Ox containing a units, β a vector along Oy containing b units,

$$\alpha = ai, \quad \beta = bj, \quad \text{and} \quad \alpha\beta = abij.$$

In the same way

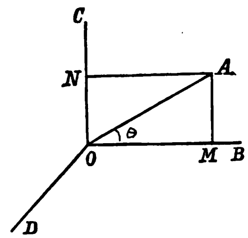
$$a^2 = ai \,.\, ai = a^2 i^2 = -a^2,$$

or the square of a *vector* is the square of the corresponding *line* with the negative sign.

Seeing therefore the facility with which we can introduce tensors whenever wanted, we may direct our principal attention, as far as multiplication is concerned, to unit vectors.

21. We proceed then next to find the product $\alpha\beta$, when α and β are vectors not at right angles to one another.

1. Let α, β be unit vectors.



Let $OA = \alpha$, $OB = \beta$.

Take $OC = \gamma$, a unit vector perpendicular to OB and in the plane BOA. Take also DO or DO produced $= \epsilon$, a unit vector perpendicular to the plane BOA.

Draw AM, AN perpendicular to OB, OC, and let the angle $BOA = \theta$; then

$$\begin{aligned} \text{vector } OA &= OM + MA = OM + ON \quad \text{(Art. 1)} \\ &= \text{part of } OB + \text{part of } OC \quad \text{(Art. 3)}. \end{aligned}$$

Now it is evident that OM as a line is that part of OB which is represented by the multiplier $\cos\theta$, or $OM = OB\cos\theta$, and similarly that $ON = OC\sin\theta$: consequently (Art. 3) the same applies to them as vectors; i. e.

$$\text{vector } OM = \beta\cos\theta, \;\; \text{vector } ON = \gamma\sin\theta;$$

$$\therefore\; \alpha = \beta\cos\theta + \gamma\sin\theta,$$

and

$$\begin{aligned} \alpha\beta &= (\beta\cos\theta + \gamma\sin\theta)\,\beta \\ &= \beta^2\cos\theta + \gamma\beta\sin\theta. \end{aligned}$$

But

$$\beta^2 = -1 \;\; (19.\ 1),$$
$$\gamma\beta = \epsilon \;\; (17);$$

[Observe that γ, β and ϵ of the present Article correspond to j, i and $-k$ of Art. 17.]

$$\therefore\; \alpha\beta = -\cos\theta + \epsilon\sin\theta.$$

2. If α, β are not unit vectors, but contain $T\alpha$ and $T\beta$ units respectively, we have at once, by the principle laid down in Art. 20,

$$\alpha\beta = T\alpha T\beta\,(-\cos\theta + \epsilon\sin\theta).$$

3. It thus appears that the product of two vectors α, β not at right angles to each other consists of two distinct parts, a numerical quantity and a vector perpendicular to the plane of α, β. The former of these Sir William Hamilton terms the SCALAR part, the latter the VECTOR part. We may now write

$$\alpha\beta = S\alpha\beta + V\alpha\beta,$$

where S is read scalar, V vector: and we find

$$S\alpha\beta = -T\alpha T\beta\cos\theta,$$
$$V\alpha\beta = T\alpha T\beta\,\epsilon\sin\theta.$$

4. The coefficient of ϵ in $V\alpha\beta$ is the area of the parallelogram whose sides are equal and parallel to the lines of which α, β are the vectors.

22. To obtain $\beta\alpha$ we have, α and β being unit vectors,

$$\alpha = \beta\cos\theta + \gamma\sin\theta;$$
$$\begin{aligned}\therefore\ \beta\alpha &= \beta(\beta\cos\theta + \gamma\sin\theta)\\ &= \beta^2\cos\theta + \beta\gamma\sin\theta\\ &= -\cos\theta - \epsilon\sin\theta \quad \text{(Art. 19, 1 and 18)};\end{aligned}$$

therefore generally

$$\beta\alpha = T\alpha T\beta\,(-\cos\theta - \epsilon\sin\theta).$$

It is scarcely necessary to remark that whilst γ operating on β turns it inwards from OB to DO produced, β operating on γ turns it outwards from OC to OD, causing it to become $-\epsilon$.

We have therefore

1. $S\alpha\beta = S\beta\alpha.$
2. $V\alpha\beta = -V\beta\alpha.$
3. $\alpha\beta + \beta\alpha = 2S\alpha\beta.$
4. $\alpha\beta - \beta\alpha = 2V\alpha\beta.$

5. $(\alpha+\beta)^2 = (\alpha+\beta)(\alpha+\beta)$
$$= \alpha^2 + \alpha\beta + \beta\alpha + \beta^2$$
$$= \alpha^2 + 2S\alpha\beta + \beta^2.$$

6. $(\alpha-\beta)^2 = \alpha^2 - 2S\alpha\beta + \beta^2.$

7. If α, β are at right angles to each other, $S\alpha\beta = 0$, and conversely.

8. $V\alpha\beta$ is a vector in the direction perpendicular to the plane which passes through α, β.

9. $\alpha^2\beta^2 = \alpha\beta \,.\, \beta\alpha$ because β^2 is a scalar;
$$\therefore\ \alpha^2\beta^2 = (S\alpha\beta + V\alpha\beta)(S\alpha\beta - V\alpha\beta)$$
$$= (S\alpha\beta)^2 - (V\alpha\beta)^2.$$

Note. $\alpha^2\beta^2$ must not be confounded with $(\alpha\beta)^2$.

23. Before proceeding further it is desirable we should work out a few simple Examples.

Ex. 1. *To express the cosine of an angle of a triangle in terms of the sides.*

Let ABC be a triangle; and retaining the usual notation of Trigonometry, let
$$CB = \alpha,\ CA = \beta;$$
then $$(\text{vector } AB)^2 = (\alpha-\beta)^2$$
$$= \alpha^2 - 2S\alpha\beta + \beta^2 \quad (22.\ 6),$$
or, changing all the signs to pass from vectors to lines (20) and applying 21. 3,
$$c^2 = a^2 - 2ab\cos C + b^2.$$

Ex. 2. *To express the relations between the sides and opposite angles of a triangle.*

Let $$CB = \alpha,\ CA = \beta,\ BA = \gamma.$$

Then $CB + BA = CA$ gives
$$\alpha + \gamma = \beta,$$
$$\alpha = \beta - \gamma;$$
$$\therefore\ \alpha^2 = \alpha(\beta-\gamma) = \alpha\beta - \alpha\gamma.$$

Take the vectors of each side.

Now $V\alpha^2 = 0$, for $\alpha^2 = -a^2$ has no vector part,

$$\therefore \; V\alpha\beta = V\alpha\gamma;$$

$$\text{i. e. (21. 3)} \;\; ab\epsilon \sin C = ac\epsilon \sin B,$$

$$\text{or} \;\; b \sin C = c \sin B;$$

$$\text{i. e.} \;\; b : c :: \sin B : \sin C.$$

Ex. 3. *The sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of the sides.*

Retaining the notation and figure of Ex. 1, Art. 7,

$$CB = \alpha + \beta,$$

$$DA = \alpha - \beta;$$

$$\therefore \; CB^2 + DA^2 = 2\alpha^2 + 2\beta^2,$$

and, changing all the signs, we get (20) for the corresponding *lines*,

$$\begin{aligned} CB^2 + DA^2 &= 2CA^2 + 2AB^2 \\ &= CA^2 + AB^2 + BD^2 + DC^2. \end{aligned}$$

Ex. 4. *Parallelograms upon the same base and between the same parallels are equal.*

It is necessary to remind the reader of what we have already stated, that examples such as this are given for illustration only. We *assume* that the area of the parallelogram is the product of two adjacent sides and the sine of the contained angle.

Adopting the figure of Euclid I. 35 and writing $TV\beta\alpha$ as the tensor multiplier of $V\beta\alpha$ so as to drop the vector ϵ on both sides; we have, calling BA, α; BC, β;

$$\begin{aligned} BE &= BA + AE \\ &= \alpha + x\beta; \end{aligned}$$

$$\therefore \; V.\beta(\alpha + x\beta) = V(BC.BE),$$

$$\text{i.e.} \;\; V\beta\alpha = V(BC.BE),$$

remembering that $x\beta^2$ has no vector part.

Hence $$T.V\beta\alpha = T(BC.BE),$$

$$\text{i.e.} \;\; BC.BA \sin ABC = BC.BE \sin EBC \;(21.\ 3),$$

which proves the proposition.

Ex. 5. *On the sides AB, AC of a triangle are constructed any two parallelograms $ABDE$, $ACFG$: the sides DE, FG are produced to meet in H. Prove that the sum of the areas of the parallelograms $ABDE$, $ACFG$ is equal to the area of the parallelogram whose adjacent sides are respectively equal and parallel to BC and AH.*

Let $$BA = \alpha,\ AE = \beta,\ AC = \gamma,\ GA = \delta,$$

then $$AH = \beta + x\alpha, \text{ and } AH = -\delta - y\gamma;$$

$$\therefore\ V\alpha AH = V\alpha\beta \text{ and } V\gamma AH = -V\gamma\delta$$
$$= V\delta\gamma\ (22.\ 2),$$

hence $$V(\alpha + \gamma)AH = V\alpha\beta + V\delta\gamma,$$

i.e. (21. 4), the parallelogram whose sides are parallel and equal to BC, AH, equals the two parallelograms whose sides are parallel and equal to BA, AE; GA, AC respectively.

[The reader is requested to notice that the *order* GA, AC is the same as the order BA, AE, and BA, AH: so that the vector ϵ is common to all.]

Ex. 6. *If O be any point whatever either in the plane of the triangle ABC or out of that plane, the squares of the sides of the triangle fall short of three times the squares of the distances of the angular points from O, by the square of three times the distance of the mean point from O.*

Let $$OA = \alpha,\ OB = \beta,\ OC = \gamma,$$

then (Art. 14), $$OG = \frac{1}{3}(\alpha + \beta + \gamma),$$

or $$\alpha^2 + \beta^2 + \gamma^2 + 2S(\alpha\beta + \beta\gamma + \gamma\alpha) = 9OG^2.$$

Now $$AB = \beta - \alpha,\ BC = \gamma - \beta,\ CA = \alpha - \gamma,$$

$$\therefore\ AB^2 + BC^2 + CA^2 = 2(\alpha^2 + \beta^2 + \gamma^2) - 2S(\alpha\beta + \beta\gamma + \gamma\alpha)$$
$$= 3(\alpha^2 + \beta^2 + \gamma^2) - 9OG^2,$$

and the lines

$$AB^2 + BC^2 + CA^2 = 3(OA^2 + OB^2 + OC^2) - (3OG)^2.$$

Ex. 7. *The sum of the squares of the distances of any point O from the angular points of the triangle exceeds the sum of the*

squares of its distances from the middle points of the sides by the sum of the squares of half the sides.

Retaining the notation of the last example, and the figure of Ex. 4, Art. 7,

$$OD=\frac{1}{2}(\beta+\gamma),\ OE=\frac{1}{2}(\gamma+\alpha),\ OF=\frac{1}{2}(\alpha+\beta);$$

$$\begin{aligned}\therefore\ 4(OD^2+OE^2+OF^2)&=2(\alpha^2+\beta^2+\gamma^2)+2S(\alpha\beta+\beta\gamma+\gamma\alpha)\\&=\alpha^2+\beta^2+\gamma^2+9OG^2\\&=4(\alpha^2+\beta^2+\gamma^2)-(AB^2+BC^2+CA^2);\end{aligned}$$

$$\therefore \text{ as lines } OD^2+OE^2+OF^2+\frac{AB^2+BC^2+CA^2}{4}=OA^2+OB^2+OC^2.$$

Ex. 8. *The squares of the sides of any quadrilateral exceed the squares of the diagonals by four times the square of the line which joins the middle points of the diagonals.*

Retaining the figure and notation of Ex. 8, Art. 7, we have squares of sides as vectors

$$\begin{aligned}&=\alpha^2+(\beta-\alpha)^2+(\gamma-\beta)^2+\gamma^2\\&=2(\alpha^2+\beta^2+\gamma^2)-2S(\alpha\beta+\beta\gamma),\end{aligned}$$

and squares of diagonals

$$\begin{aligned}&=\beta^2+(\gamma-\alpha)^2\\&=\alpha^2+\beta^2+\gamma^2-2S\alpha\gamma;\end{aligned}$$

therefore the former sum exceeds the latter by

$$\begin{aligned}\alpha^2+\beta^2+\gamma^2&-2S\alpha\beta-2S\beta\gamma+2S\alpha\gamma\\&=(\alpha+\gamma-\beta)^2\\&=4\left(\frac{\alpha+\gamma}{2}-\frac{\beta}{2}\right)^2\\&=4(OQ-OP)^2\\&=4PQ^2.\end{aligned}$$

Therefore as lines the same is true.

Note. The points A, B, C, D may be in different planes.

Ex. 9. *Four times the squares of the distances of any point whatever from the angular points of a quadrilateral are equal to the sum of the squares of the sides, the squares of the diagonals and the square of four times the distance of the point from the mean point of the figure.*

With the notation of Art. 14, and the figure of Ex. 7, Art. 7, we have

squares of the sides + squares of the diagonals

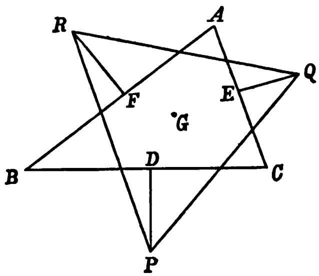
$$= (\beta - \alpha)^2 + (\gamma - \beta)^2 + (\delta - \gamma)^2 + (\alpha - \delta)^2 + (\gamma - \alpha)^2 + (\delta - \beta)^2$$
$$= 3\,(\alpha^2 + \beta^2 + \gamma^2 + \delta^2) - 2S\,(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta).$$

Now (Art. 14) $\quad (\alpha + \beta + \gamma + \delta)^2 = (4OX)^2$;

$$\therefore\ (4OX)^2 + \text{squares of sides} + \text{squares of diagonals}$$
$$= 4\,(OA^2 + OB^2 + OC^2 + OD^2).$$

Ex. 10. *The lines which join the mean points of three equilateral triangles described outwards on the three sides of any triangle form an equilateral triangle whose mean point is the same as that of the given triangle.*

Let P, Q, R be the mean points of the equilateral triangles on BC, CA, AB; $PD = \alpha$, $DC = \beta$, $CE = \gamma$, $EQ = \delta$; and let the sides of the triangle ABC be $2a$, $2b$, $2c$.



$$\therefore PQ^2 = (\alpha + \beta + \gamma + \delta)^2$$
$$= \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + 2S\alpha\beta + 2S\alpha\gamma + 2S\alpha\delta$$
$$+ 2S\beta\gamma + 2S\beta\delta + 2S\gamma\delta.$$

Changing all the signs and observing that

$$S\alpha\beta = 0,\ \ S\alpha\gamma = -ab\sin C,\ \&c.$$

we have (writing the results in the same order),

$$\text{line } PQ^2 = \frac{a^2}{3} + a^2 + b^2 + \frac{b^2}{3} + 0$$

$$+\frac{2}{\sqrt{3}}ab\sin C + \frac{2}{3}ab\cos C - 2ab\cos C + \frac{2}{\sqrt{3}}ab\sin C + 0$$

$$= \frac{4}{3}(a^2 + b^2 - ab\cos C) + \frac{4}{\sqrt{3}}ab\sin C$$

$$= \frac{2}{3}(a^2 + b^2 + c^2) + \frac{2}{\sqrt{3}}\text{ area of } ABC,$$

which being symmetrical in a, b, c proves that PQR is equilateral.

Again, G being the mean point of ABC,

$$PG = PD + DG = \alpha + \frac{\beta}{3} + \frac{2\gamma}{3},$$

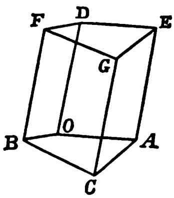
$$\therefore\ PG^2 = \alpha^2 + \frac{\beta^2}{9} + \frac{4\gamma^2}{9} + \frac{2}{3}S\alpha\beta + \frac{4}{3}S\alpha\gamma + \frac{4}{9}S\beta\gamma,$$

$$\text{and line } PG^2 = \frac{a^2}{3} + \frac{a^2}{9} + \frac{4b^2}{9} + \frac{4}{3\sqrt{3}}ab\sin C - \frac{4}{9}ab\cos C$$

$$= \frac{2}{9}(a^2 + b^2 + c^2) + \frac{2}{3\sqrt{3}}\text{ area } ABC;$$

$$\therefore\ PG = QG = RG;$$

and G is the mean point of the equilateral triangle PQR.



Ex. 11. *In any quadrilateral prism, the sum of the squares of the edges exceeds the sum of the squares of the diagonals by eight times the square of the straight line which joins the points of intersection of the two pairs of diagonals.*

Let $OA = \alpha$, $OB = \beta$, $OC = \gamma$, $OD = \delta$;
sum of squares of edges =

$$2\{\alpha^2 + \beta^2 + (\gamma - \alpha)^2 + (\gamma - \beta)^2 + 2\delta^2\}$$

$$= 2\{2\alpha^2 + 2\beta^2 + 2\gamma^2 + 2\delta^2 - 2S\alpha\gamma - 2S\beta\gamma\},$$

sum of squares of diagonals

$$=(\delta+\gamma)^2+(\delta-\gamma)^2+(\delta+\alpha-\beta)^2+(\delta+\beta-\alpha)^2$$
$$=2\{\alpha^2+\beta^2+\gamma^2+2\delta^2-2S\alpha\beta\}.$$

Also $$\frac{1}{2}OG=\frac{1}{2}(\delta+\gamma)$$

$=$ vector to the point of bisection of CD, and therefore to the point of intersection of OG, CD, and vector from O to the point of bisection of AF, as also to that of BE, and therefore to the intersection of AF, BE

$$=\frac{1}{2}(\delta+\alpha+\beta),$$

hence vector which joins the points of intersection of diagonals

$$=\frac{1}{2}(\alpha+\beta-\gamma),$$

eight times square of this vector

$$=2(\alpha^2+\beta^2+\gamma^2+2S\alpha\beta-2S\alpha\gamma-2S\beta\gamma),$$

which, added to the sum of the squares of the diagonals, makes up the sum of the squares of the edges.

24. Definition. We define the quotient or fraction $\frac{\beta}{\alpha}$, where α and β are unit vectors, to be such that when it operates *on* α it produces β or $\frac{\beta}{\alpha}.\alpha=\beta$. This form of the definition enables us to strike out α by a dash made in the direction of ordinary writing, thus $\frac{\beta}{\alpha}.\alpha=\beta$, $\frac{\beta}{\alpha}$ is therefore that multiplier which, operating on α, or on $\beta\cos\theta+\gamma\sin\theta$ (21), produces β.

Now $\cos\theta+\epsilon\sin\theta$ operating on $\beta\cos\theta+\gamma\sin\theta$ produces

$$\beta\cos^2\theta+(\gamma+\epsilon\beta)\sin\theta\cos\theta+\epsilon\gamma\sin^2\theta.$$

But a glance at the figure (Art. 21) will shew that

$$\epsilon\beta=-\gamma,$$

and $$\epsilon\gamma=\beta;$$

$\therefore$ $\cos\theta + \epsilon\sin\theta$ operating on $\beta\cos\theta + \gamma\sin\theta$ produces β;

hence $$\frac{\beta}{\alpha} = \cos\theta + \epsilon\sin\theta.$$

It may be worth while to exhibit another demonstration of this proposition: thus

$$\frac{\beta}{\alpha}.\alpha\beta = \beta.\beta \text{ (by the associative law)} = -1.(19.1).$$

$$\text{i.e. } (21.1)\ \frac{\beta}{\alpha}.(-\cos\theta + \epsilon\sin\theta) = -1.$$

Now $$(\cos\theta + \epsilon\sin\theta)(-\cos\theta + \epsilon\sin\theta)$$
$$= -\cos^2\theta - \sin^2\theta$$
$$= -1;$$
$$\therefore \frac{\beta}{\alpha} = \cos\theta + \epsilon\sin\theta.$$

COR. $\frac{\beta}{\alpha} = -\beta\alpha$ (by 22).

25. 1. DEFINITION. Still retaining α, β as unit vectors, since $\frac{\beta}{\alpha}$ operating on α causes it to become β, it may be defined as a VERSOR acting as if its axis were along OD (Fig. Art. 21). By comparing the result of that article with the definitions of Art. 17, it is clear that $\frac{\beta}{\alpha}$ or $\cos\theta + \epsilon\sin\theta$ is an operator of the same character as $-k$ or ϵ (as we have now called the corresponding unit vector); with this difference only, that whereas $-k$ or ϵ as an operator would turn α through a right angle, $\cos\theta + \epsilon\sin\theta$ turns it, in the same direction, only through the angle θ: $\cos\theta + \epsilon\sin\theta$ is then the *versor* through the angle θ.

2. If α, β are not unit vectors, the considerations already advanced render it evident that

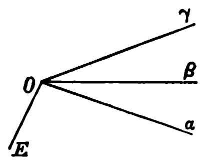
$$\frac{\beta}{\alpha} = \frac{T\beta}{T\alpha}(\cos\theta + \epsilon\sin\theta).$$

Now $\frac{T\beta}{T\alpha}$ is itself of the nature of a tensor, for it is a numerical quantity, hence $\frac{\beta}{\alpha}$ is the product of a tensor and a versor.

26. By comparing the last Article with Art. 22 it appears that generally the product or quotient of two vectors may be expressed as the product of a tensor and a versor. This product Sir W. Hamilton names a QUATERNION.

COR. It is evident that a quaternion is also the sum of a scalar and a vector.

27. (1) If α, β, γ are unit vectors in the same plane, ϵ a unit vector perpendicular to that plane; we have seen that $\frac{\beta}{\alpha}$ operating on α turns it round about ϵ as an axis to bring it into the position β. If now $\frac{\gamma}{\beta}$ be a second operator about the same axis in the same direction acting on β, it will bring it into the position γ. But it is evident that $\frac{\gamma}{\alpha}$ acting on α would at once have brought it into the position γ. This is equivalent to the fact that $\frac{\gamma}{\beta}\cdot\frac{\beta}{\alpha}=\frac{\gamma}{\alpha}$; or in another form (Art. 24) that



$$(\cos\phi+\epsilon\sin\phi)(\cos\theta+\epsilon\sin\theta)=\cos(\theta+\phi)+\epsilon\sin(\theta+\phi).$$

From this it is evident that the results of Demoivre's Theorem apply to the form $\cos\theta+\epsilon\sin\theta$.

Further, it is evident that since $\cos\theta+\epsilon\sin\theta$ operating with ϵ as its axis, turns a vector through the angle θ, whilst ϵ itself acting in the same direction turns it through a right angle, $\cos\theta+\epsilon\sin\theta$ is *part* of the operation designated by ϵ, viz. that part which bears to the whole the proportion that θ bears to a right angle.

(2) Remembering then that the operations are of the nature of multiplication, it becomes evident that $\cos\theta+\epsilon\sin\theta$ as an operator may be abbreviated by $\epsilon^{\frac{\theta}{\frac{\pi}{2}}}$ or $\epsilon^{\frac{2\theta}{\pi}}$.

And since

$$(\cos\theta+\epsilon\sin\theta)(\cos\phi+\epsilon\sin\phi)=\cos(\theta+\phi)+\epsilon\sin(\theta+\phi),$$

we shall have

$$\epsilon^{\frac{2\theta}{\pi}} \cdot \epsilon^{\frac{2\phi}{\pi}} = \epsilon^{\frac{2}{\pi}(\theta+\phi)},$$

or the *law of indices* is applicable to this operator.

(3) Now we have already seen (19. 1) that $\epsilon^2 = -1$;

$$\therefore\ \epsilon^4 = +1.$$

Conversely, if $\epsilon^n = \pm \epsilon$, n must be an odd number; if $\epsilon^n = -1$, n must be an odd multiple of 2; and if $\epsilon^n = +1$, n must be an even multiple of 2.

(4) When α, β are not units, the introduction of the corresponding tensor can be at once effected.

We conclude that a quaternion may be expressed as the *power* of a vector, to which the algebraic definition of an index is applicable.

28. *Reciprocals* of quaternions—unit vectors.

1. Since $$\alpha . \alpha = \alpha^2 = -1,$$

and $$\frac{1}{\alpha} . \alpha = 1 \text{ (Def. Art. 24)}$$

$$= -\alpha . \alpha;$$

$$\therefore\ \frac{1}{\alpha} = -\alpha, \text{ or } \alpha^{-1} = -\alpha;$$

or the reciprocal of a unit vector is a unit vector in the opposite direction.

2. Again, $$\alpha . \frac{1}{\alpha} = \alpha(-\alpha) = 1 = \frac{1}{\alpha} . \alpha;$$

or a vector is commutative with its reciprocal.

3. If q be a versor $\left(\text{say } \cos\theta + \epsilon \sin\theta, \text{ or } \frac{\beta}{\alpha}\right)$,

$$\frac{1}{q} . q = 1 \text{ (Def. extended).}$$

Now $$\frac{\beta}{\alpha} = q;$$

$$\therefore\ \beta = q\alpha, \text{ by operating } on\ \alpha.$$

Also
$$\frac{\alpha}{\beta} = \frac{1}{q},$$

$$\alpha = \frac{1}{q}\beta, \text{ by operating on } \beta,$$

and
$$\beta = q\alpha = q \cdot \frac{1}{q}\beta;$$

$$\therefore \; q \cdot \frac{1}{q} = 1 = \frac{1}{q} \cdot q,$$

or q and $\frac{1}{q}$ are commutative.

This is perhaps better demonstrated by observing that

$$\frac{\beta}{\alpha} \cdot \frac{\alpha}{\beta} = \frac{\beta}{\beta} = 1;$$

or that if
$$\frac{\beta}{\alpha} = \cos\theta + \epsilon\sin\theta,$$

then must
$$\frac{\alpha}{\beta} = \cos\theta - \epsilon\sin\theta;$$

factors which are from their very nature commutative.

When the versors are not units the tensors can be introduced as mere multipliers without affecting the versor conclusions.

29. We present one or two examples of quaternion division.

Ex. 1. T*o express sin* $(\theta + \phi)$ *and cos* $(\theta + \phi)$ *in terms of sines and cosines of* θ *and* ϕ.

α, β, γ being unit vectors in the same plane (Fig. Art. 27), we have

$$\frac{\beta}{\alpha} = \cos\theta + \epsilon\sin\theta,$$

$$\frac{\gamma}{\beta} = \cos\phi + \epsilon\sin\phi,$$

$$\frac{\gamma}{\alpha} = \cos(\theta + \phi) + \epsilon\sin(\theta + \phi).$$

But $$\frac{\gamma}{\alpha} = \frac{\gamma}{\beta} \cdot \frac{\beta}{\alpha};$$

$$\therefore \cos(\theta + \phi) + \epsilon \sin(\theta + \phi) = (\cos\theta + \epsilon \sin\theta)(\cos\phi + \epsilon \sin\phi);$$

whence multiplying out and equating, we have

$$\sin(\theta + \phi) = \sin\theta \cos\phi + \cos\theta \sin\phi,$$

$$\cos(\theta + \phi) = \cos\theta \cos\phi - \sin\theta \sin\phi.$$

COR. If the action of the versors be in opposite directions, β lying beyond γ, we have (Art. 28)

$$\frac{\alpha}{\gamma} = \cos(\theta - \phi) - \epsilon \sin(\theta - \phi).$$

But $$\frac{\beta}{\gamma} = \cos\phi + \epsilon \sin\phi,$$

$$\frac{\alpha}{\beta} = \cos\theta - \epsilon \sin\theta;$$

$$\therefore \frac{\alpha}{\gamma} = \frac{\alpha}{\beta} \cdot \frac{\beta}{\gamma} \text{ gives}$$

$$\cos(\theta - \phi) - \epsilon \sin(\theta - \phi) = (\cos\theta - \epsilon \sin\theta)(\cos\phi + \epsilon \sin\phi),$$

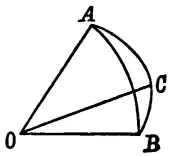
whence $$\sin(\theta - \phi) = \sin\theta \cos\phi - \cos\theta \sin\phi,$$

$$\cos(\theta - \phi) = \cos\theta \cos\phi + \sin\theta \sin\phi.$$

EX. 2. *To find the cosine of the angle of a spherical triangle in terms of the sides.*

Let α, β, γ be unit vectors OA, OB, OC not in the same plane, then

$$\frac{\beta}{\gamma} = \frac{\beta}{\alpha} \cdot \frac{\alpha}{\gamma};$$



i. e. taking the scalars of each side,

$$\cos a = \cos c \cos b + S \cdot \left(V\frac{\beta}{\alpha} \cdot V\frac{\alpha}{\gamma}\right).$$

Now $SV\frac{\beta}{\alpha}V\frac{\alpha}{\gamma}$ is $\sin c \sin b \times$ cosine of the angle between perpendiculars to the planes AB, AC, and is therefore

$$\sin b \sin c \cos A;$$

$$\therefore \cos a = \cos c \cos b + \sin c \sin b \cos A.$$

The reader will observe that in accordance with the results of Art. 21, the sign of the term involving $\cos A$ is $+$, seeing that it is in fact $-$cosine (supplement of A).

Ex. 3. *The angles of a triangle are together equal to two right angles.*

What we shall prove in fact is that the exterior angles formed by producing the sides in the same direction are equal to four right angles.

Let unit vectors along BC, CA, AB be α, β, γ; and let the exterior angles formed by producing BC, CA, AB be θ, ϕ, ψ; then

$$\epsilon^{\frac{2\theta}{\pi}}\alpha = \beta \quad (27.\ 1),$$

$$\epsilon^{\frac{2\phi}{\pi}}\beta = \gamma,$$

$$\epsilon^{\frac{2\psi}{\pi}}\gamma = \alpha;$$

$$\therefore\ \epsilon^{\frac{2\phi}{\pi}} \cdot \epsilon^{\frac{2\theta}{\pi}}\alpha = \epsilon^{\frac{2\phi}{\pi}}\beta = \gamma,$$

and
$$\epsilon^{\frac{2\psi}{\pi}} \cdot \epsilon^{\frac{2\phi}{\pi}} \cdot \epsilon^{\frac{2\theta}{\pi}}\alpha = \epsilon^{\frac{2\psi}{\pi}}\gamma = \alpha,$$

so that
$$\epsilon^{\frac{2\psi}{\pi}} \cdot \epsilon^{\frac{2\phi}{\pi}} \cdot \epsilon^{\frac{2\theta}{\pi}} = 1,$$

or
$$\epsilon^{\frac{2}{\pi}(\theta+\phi+\psi)} = 1 \quad (27.\ 2).$$

Hence (27. 3), $\frac{2}{\pi}(\theta+\phi+\psi)$ is an even multiple of 2. The first value is 4;

$$\therefore\ \theta+\phi+\psi = 2\pi,$$

or the exterior angles of a triangle are equal to four right angles.

It will be seen that the demonstration here given is of the nature of that given by Prof. Thomson in the Notes to his Euclid.

Ex. 4. *In the figure of Euclid* I. 47 *the three lines AL, BK, CF meet in a point.*

Let $BC = \alpha$, $CA = \beta$, $AB = \gamma$; the sides being as usual denoted by a, b, c.

Let i be the vector which turns another negatively through a right angle in the plane of the paper, so that

$$BD = i\alpha,\ CK = i\beta,\ AG = i\gamma.$$

If BK, AL meet in O,

$$BO = xBK = x(\alpha + i\beta),$$

and

$$BO = BA + AO = BA + yBD$$

$$= -\gamma + yi\alpha;$$

$$\therefore\ x(\alpha + i\beta) = -\gamma + yi\alpha,$$

$$xS\alpha(\alpha + i\beta) = -S\alpha\gamma,$$

$$x = -\frac{S\alpha\gamma}{S\alpha(\alpha + i\beta)} = \frac{ac\cos B}{a^2 + ab\sin C}$$

$$= \frac{c^2}{a^2 + bc},$$

and

$$xS\alpha\beta = ySi\alpha\beta;$$

$$\therefore\ y = \frac{b}{c}x = \frac{bc}{a^2 + bc},$$

which being symmetrical in b and c shews that CF, AL intersect in the same point in which BK, AL intersect.

COR. Since

$$\frac{BO}{BK} = \frac{c^2}{a^2 + bc},$$

we have

$$\frac{CO}{CF} = \frac{b^2}{a^2 + bc};$$

also

$$\frac{AO}{BD} = \frac{bc}{a^2 + bc};$$

$$\therefore \frac{AO}{BD} + \frac{BO}{BK} + \frac{CO}{CF} = \frac{c^2 + b^2 + bc}{a^2 + bc} = 1.$$

Ex. 5. *If ABCD be a quadrilateral inscribed in a circle;*

$$AB = \alpha,\ BC = \beta,\ CD = \gamma,\ DA = \delta;$$

then
$$\alpha\beta\gamma = \frac{T\alpha T\beta T\gamma}{T\delta}\delta.$$

Let unit vectors along AB, BC, CD, DA be α', β', γ', δ'; and let the exterior angles at B and D be θ and ϕ respectively; then

$$\begin{aligned} \alpha'\beta'\gamma' &= (-\cos\theta + \epsilon\sin\theta)\,\gamma' \quad (21.\ 1) \\ &= (\cos\phi + \epsilon\sin\phi)\,\gamma' \\ &= \delta' \quad (25.\ 1); \end{aligned}$$

therefore, introducing the tensors,

$$\alpha\beta\gamma = \frac{T\alpha T\beta T\gamma}{T\delta}\delta.$$

Conjugate Quaternions.

30. If we designate by q the expression $-\cos\theta + \epsilon\sin\theta$, we have seen that it may be regarded as a *versor* through an angle θ in a certain direction. Now if we write $-\theta$ in place of θ in this expression it assumes the form $-\cos\theta - \epsilon\sin\theta$, which must on the same hypotheses be regarded a versor through the angle θ in the contrary direction.

When the quaternion is completed by the introduction of a tensor Tq, if we retain the same tensor to both forms of the versor, we have Sir W. Hamilton's *conjugate* quaternion defined thus: The conjugate of a quaternion q, written Kq, has the same tensor, plane and angle as q has, only the angle is taken in the reverse way.

The analogy between q and Kq is precisely the same as that which exists between the two forms

$$R\,(\cos\phi + \sqrt{-1}\,\sin\phi) \text{ and } R\,(\cos\phi - \sqrt{-1}\,\sin\phi);$$

and as the product of the latter form is R^2, so the multiplication of the former produces $(Tq)^2$.

If we put $$q = Sq + Vq,$$

we shall have $$Kq = Sq - Vq,$$

and $$qKq = (Sq)^2 + (TVq)^2,$$

for $$(Vq)^2 = -(TVq)^2, \text{ Art. 20.}$$

It is almost self-evident that, since the change of order of multiplication of two vectors produces no other change than that of the sign of the vector part of the product (22),

$$K(qr) = KrKq,$$

q and r occurring in a changed order.

The following is a demonstration.

Let $$q = Tq(-\cos\theta + \alpha\sin\theta),$$
$$r = Tr(-\cos\phi + \beta\sin\phi),$$

α and β being unit vectors; then

$$qr = TqTr(\cos\theta\cos\phi - \alpha\sin\theta\cos\phi - \beta\cos\theta\sin\phi + \alpha\beta\sin\theta\sin\phi),$$

$$\begin{aligned} KrKq &= TqTr(-\cos\phi - \beta\sin\phi)(-\cos\theta - \alpha\sin\phi) \\ &= TqTr(\cos\theta\cos\phi + \alpha\sin\theta\cos\phi + \beta\cos\theta\sin\phi + \beta\alpha\sin\theta\sin\phi). \end{aligned}$$

Now observing that $\beta\alpha$ has the same scalar part with $\alpha\beta$, but the vector part with a contrary sign, we see that the two expressions for qr and for $KrKq$ likewise have the same scalar part, but that their vector parts have contrary signs.

Hence $$K(qr) = KrKq.$$

(See Tait, § 79 et sq.)

31. We propose, in this Article, to give and interpret one or two formulæ, relating to three or more vectors, which are indispensable to our progress, reserving to a separate Chapter the demonstration and application of other formulæ, the value of which the reader can hardly as yet be expected to understand.

1. To express $S.\alpha\beta\gamma$ geometrically.

First suppose α, β, γ to be unit vectors OA, OB, OC.

Let $AOB=\theta$, and the angle which OC makes with the plane $AOB=\phi$; then since

$$\alpha\beta = -\cos\theta + \epsilon\sin\theta \text{ (Art. 21)},$$

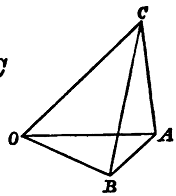
where ϵ is perpendicular to the plane AOB,

$$S.\alpha\beta\gamma = S(-\cos\theta+\epsilon\sin\theta)\gamma$$
$$= S\epsilon\gamma\sin\theta.$$

Now $S\epsilon\gamma = -\cos$. angle between ϵ and γ

$= -\sin$. angle between plane AOB and OC

$= -\sin\phi$;



$\therefore\ S.\alpha\beta\gamma = -\sin\phi\sin\theta.$

Next if α, β, γ are not units, but have respectively the lengths $T\alpha$, $T\beta$, $T\gamma$, or a, b, c; we shall have

$$S.\alpha\beta\gamma = -abc\sin\theta\sin\phi.$$

But $ab\sin\theta$ is the area of the parallelogram of which the adjacent sides are a, b; and $c\sin\phi$ is the perpendicular from C on the plane of the parallelogram;

$\therefore\ -S.\alpha\beta\gamma = ab\sin\theta.c\sin\phi$

= volume of parallelepiped of which three conterminous edges are OA, OB, OC.

2. From the nature of the case, no change of order amongst the vectors α, β, γ can make any change in the value (apart from the sign) of the scalar of the product of the three vectors; for it will in every case produce the volume of the same parallelepiped.

$$\therefore\ S.\alpha\beta\gamma = \pm S.\gamma\alpha\beta = \pm S.\alpha\gamma\beta, \text{ \&c.}$$

COR. 1. The volume of the triangular pyramid, of which OA, OB, OC are conterminous edges is $-\dfrac{1}{6}S.\alpha\beta\gamma$.

COR. 2. If α, β, γ are in the same plane, $\phi = 0$;

$$\therefore\ S.\alpha\beta\gamma = 0.$$

Conversely, if $S.\alpha\beta\gamma = 0$, none of the vectors α, β, γ being themselves 0, we must have either $\theta = 0$ or $\phi = 0$; hence in either case the three vectors are co-planar.

3. Since $V\alpha\beta = \gamma'$ (21. 3), a vector perpendicular to the plane OAB (Fig. of formula 2); $V\beta\gamma = \alpha'$, a vector perpendicular to the plane OBC; and since γ', α' are both perpendicular to OB, the line along which is the vector β; OB is perpendicular to the plane which passes through γ', α', and therefore (21. 3) is in the direction of $V\gamma'\alpha'$; hence

$$V(V\alpha\beta\, V\beta\gamma) = V\gamma'\alpha' = m\beta,$$

or the vector of the product of two resultant vectors, one of the constituents of each of which is the same vector, is a multiple of that vector.

4. If $OA = \alpha$, $OB = \beta$, $OD = \delta$, $OE = \epsilon$; and if the planes OAB, ODE intersect in OP; it follows, as in (3), that, $V\alpha\beta$ and $V\delta\epsilon$ being both perpendicular to OP,

$$V(V\alpha\beta V\delta\epsilon) \text{ is along } OP \text{ and is therefore} = nOP.$$

5. *Connection between the representation of the position of a point by a vector and its representation by Cartesian co-ordinates.*

If x, y, z be the perpendicular distances of a point P in space from the planes of yz, zx, xy respectively (fig. of Art. 16); i, j, k unit vectors in the directions of x, y, z; then xi is the *vector* of which the line is x (Art. 3); consequently OM along Ox, MN parallel to Oy and NP parallel to Oz, being x, y, z as co-ordinates, they are xi, yj, zk as vectors.

Now $\qquad$ vector $OP = OM + MN + NP$,

and is therefore $\qquad \rho = xi + yj + zk.$

The same method of representation is evidently applicable when the planes of reference are not mutually at right angles. If x, y, z be the co-ordinates of P referred to oblique co-ordinates; α, β, γ unit vectors parallel respectively to x, y, z; then

$$\text{vector } OP = x\alpha + y\beta + z\gamma.$$

COR. When x, y, z are at right angles to one another

$$\rho = xi + yj + zk$$

gives
$$Si\rho = -x, \; Sj\rho = -y, \; Sk\rho = -z;$$
$$\therefore \; (Si\rho)^2 + (Sj\rho)^2 + (Sk\rho)^2 = x^2 + y^2 + z^2$$
$$= OP^2.$$

EX. *To find the volume of the pyramid of which the vertex is a given point and the base the triangle formed by joining three given points in the rectangular co-ordinate axes.*

Let A, B, C be the three given points;

$$\text{line } OA = a, \; OB = b, \; OC = c;$$

x, y, z the co-ordinates of the given point P,

then
$$\text{vector } OA = ai, \; OB = bj, \; OC = ck;$$
and
$$OP = xi + yj + zk;$$
$$\therefore \; PA = OA - OP = -\{(x-a)\,i + yj + zk\},$$
$$PB = -\{xi + (y-b)\,j + zk\},$$
$$PC = -\{xi + yj + (z-c)\,k\}.$$

Now the volume of the pyramid $PABC$ is

$$-\frac{1}{6} S(PA.PB.PC) \quad (31.\ 2.\ \text{Cor. } 1)$$

$$= -\frac{1}{6} S.\{(x-a)\,i + yj + zk\}\{xi + (y-b)\,j + zk\}\{xi + yb + (z-c)\,k\}.$$

Multiplying out and observing that only terms which involve all of the three vectors i, j, k produce a scalar in the product, we get

$$(+ \text{ or } -) \text{ Vol.} = -\frac{1}{6}\{(x-a)(bz + cy - bc) - cxy - bxz\}$$

$$= \frac{1}{6} abc \left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1\right).$$

The *sign* of the result will of course depend on the position of P.

ADDITIONAL EXAMPLES TO CHAP. III.

1. If in the figure of Euclid I. 47 DF, GH, KE be joined, the sum of the squares of the joining lines is three times the sum of the squares of the sides of the triangle.

The same is true whatever be the angle A.

2. Prove that

$$4AD^2 \text{ (Art. 7, Ex. 4)} = 2(AB^2 + AC^2) - BC^2.$$

3. If P, Q, R, S be points in the sides AB, BC, CD, DA of a rectangle, such that $PQ = RS$, prove that

$$AR^2 + CS^2 = AQ^2 + CP^2.$$

4. The sum of the squares of the three sides of a triangle is equal to three times the sum of the squares of the lines drawn from the angles to the mean point of the triangle.

5. In any quadrilateral, the product of the two diagonals and the cosine of their contained angle is equal to the sum or difference of the two corresponding products for the pairs of opposite sides.

6. If a, b, c be three conterminous edges of a rectangular parallelepiped; prove that four times the square of the area of the triangle which joins their extremities is

$$= a^2b^2 + b^2c^2 + c^2a^2.$$

7. If two pairs of opposite edges of a tetrahedron be respectively at right angles, the third pair will be also at right angles.

8. Given that each edge of a tetrahedron is equal to the edge opposite to it. Prove that the lines which join the points of bisection of opposite edges are at right angles to those edges.

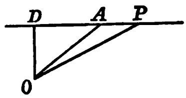
9. If from the vertex O of a tetrahedron $OABC$ the straight line OD be drawn to the base making equal angles with the faces OAB, OAC, OBC; prove that the triangles OAB, OAC, OBC are to one another as the triangles DAB, DAC, DBC.

CHAPTER IV.

THE STRAIGHT LINE AND PLANE.

32. Equations of a straight line.

1. Let β be a vector (unit or otherwise) parallel to or along the straight line; α the vector to a given point A in the line, ρ that to any point whatever P in the line, starting from the same origin O; then AP is a vector parallel to β



$$= x\beta, \text{ say,}$$

and $$OP = OA + AP$$

gives $$\rho = \alpha + x\beta \quad (1)$$

as the equation of the line.

2. Another form in which the equation of a straight line may be expressed is this: let $OA = \alpha$, $OB = \beta$ be the vectors to two given points in the line; then

$$AB = \beta - \alpha \text{ and } AP = x(\beta - \alpha);$$

$$\therefore \rho = \alpha + x(\beta - \alpha) \quad (2).$$

Of course the β of No. 2 is not that of No. 1. The first form of the equation supposes the direction of the line and the position of one point in it to be given, the second form supposes two points in it to be given.

3. A third form may be exhibited in which the perpendicular on the line from the origin is given.

Let OD perpendicular to $AP = \delta$; then

$$DP = \rho - \delta \text{ and } S\delta(\rho - \delta) = 0,$$

because OD is perpendicular to AP (22. 7);

$$\text{i. e. } S\delta\rho = C \quad (3),$$

where C is a constant.

(*Note.* In addition to this we must have the equation of the plane of the paper, in which ρ is tacitly supposed to lie. This may be written as $S\epsilon\rho = 0$.)

33. Equation of a plane.

Let P be any point in the plane, OD perpendicular to the plane; and let

$$OD = \delta, \quad OP = \rho;$$

then $$\rho - \delta = \cancel{OP}, \quad DP$$

which is in a direction perpendicular to OD;

$$\therefore S\delta(\rho - \delta) = 0,$$

$$\text{or } S\delta\rho = \delta^2,$$

$$\text{or } S\frac{\rho}{\delta} = 1.$$

COR. 1. If $S\delta\rho = C$ be the equation of a plane, δ is a vector in the direction perpendicular to the plane.

COR. 2. If the plane pass through O, ρ can have the value zero,

$$\therefore S\delta\rho = 0 \text{ is the equation.}$$

COR. 3. Since a vector can be drawn in the plane through D, parallel to any given vector in or parallel to the plane; if β be any vector in or parallel to the plane, $S\delta\beta = 0$.

34. We proceed to exhibit certain modifications of the equations of a straight line and plane, and one or two results immediately deducible from the forms of those equations.

1. To find the equation of a straight line which is perpendicular to each of two given straight lines.

Let β, γ be vectors from a given point A in the required line, and parallel respectively to the given lines.

If $OA = \alpha$ as before, then since (22. 8) $V\beta\gamma$ is a vector along the line whose equation is required; we have

$$\rho - \alpha = xV\beta\gamma,$$

$$\text{or } \rho = \alpha + xV\beta\gamma,$$

as the equation of the line.

2. To find the length of the perpendicular from the origin on a given line.

Equation (1) of Art. 32 is

$$\rho = \alpha + x\beta.$$

If now $$\rho = OD = \delta;$$

we get $$S\delta^2 = S\delta\alpha,$$

or $$-OD^2 = S\delta\alpha;$$

$$\therefore\ OD = -\frac{S\delta\alpha}{OD} = -S\alpha U\delta,$$

$U\delta$ being the unit vector perpendicular to the line.

COR. The same result is true of a plane.

3. To find the length of the perpendicular from a given point on a given plane.

Let $S\alpha\rho = C$ be the equation of the plane, γ the vector to the given point.

Then if the vector perpendicular be $x\alpha$ (33. Cor. 1),

$$\rho = \gamma + x\alpha$$

gives $$S\alpha\gamma + x\alpha^2 = C,$$

and the vector perpendicular is

$$x\alpha = -\alpha^{-1}(C - S\alpha\gamma);$$

the square of which with a $-$ sign is the square of the perpendicular.

4. To find the length of the common perpendicular to each of two given straight lines.

Let β, β_1 be unit vectors along the lines; α, α_1 vectors to given points in the lines;

$$\rho = \alpha + x\beta,$$
$$\rho_1 = \alpha_1 + x_1\beta_1,$$

the vectors to the extremities of the common perpendicular δ.

Then since δ is perpendicular to both lines, it is perpendicular to the plane which passes through two straight lines drawn parallel to them through a given point;

$$\therefore \text{ (21. 3) } \delta = yV\beta\beta_1.$$

But $$\delta = \rho - \rho_1 = \alpha + x\beta - \alpha_1 - x_1\beta_1,$$

hence $$S.\delta\beta\beta_1 = S.(\alpha - \alpha_1)\beta\beta_1;$$

$$\text{i. e. } S(yV\beta\beta_1 . \beta\beta_1) = S.(\alpha - \alpha_1)\beta\beta_1,$$

$$\text{or } y(V\beta\beta_1)^2 = S.(\alpha - \alpha_1)\beta\beta_1,$$

because $$SV\beta\beta_1 S\beta\beta_1 = 0;$$

$$\therefore y = \frac{S.(\alpha - \alpha_1)\beta\beta_1}{(V\beta\beta_1)^2},$$

whence $\delta = yV\beta\beta_1$ is known.

5. To find the equation of a plane which passes through three given points.

Let α, β, γ be the vectors of the points.

Then $\rho - \alpha$, $\alpha - \beta$, $\beta - \gamma$ are in the same plane.

$$\therefore \text{ (Art. 31. 2. Cor. 2) } S.(\rho - \alpha)(\alpha - \beta)(\beta - \gamma) = 0,$$

or $$S\rho(V\alpha\beta + V\beta\gamma + V\gamma\alpha) - S.\alpha\beta\gamma = 0$$

is the equation required.

COR. $V\alpha\beta + V\beta\gamma + V\gamma\alpha$ is a vector in the direction perpendicular to the plane; therefore (No. 3) the perpendicular vector from the origin

$$= S.\alpha\beta\gamma.(V\alpha\beta + V\beta\gamma + V\gamma\alpha)^{-1}.$$

6. To find the equation of a plane which shall pass through a given point and be parallel to each of two given straight lines.

Let γ be the vector to the given point, $\rho = \alpha + x\beta$, $\rho = \alpha_1 + x_1\beta_1$ the lines; then if lines be drawn in the required plane parallel to each of the given straight lines—these lines as vectors will be β, β_1: also $\rho - \gamma$ is a vector line in the plane;

$$\therefore\ S\,.\,\beta\beta_1(\rho - \gamma) = 0 \quad (31.\ 2.\ \text{Cor. } 2),$$

which is the equation required.

7. To find the equation of a plane which shall pass through two given points and be perpendicular to a given plane.

Let α, β be the vectors to the given points, $S\delta\rho = C$ the equation of the plane; then the three lines $\rho - \alpha$, $\alpha - \beta$, δ are vectors in the plane;

$$\therefore\ S\,.\,(\rho - \alpha)(\alpha - \beta)\,\delta = 0,$$

$$\text{or } S\,.\,\rho(\alpha - \beta)\,\delta + S\,.\,\alpha\beta\delta = 0.$$

8. *To find the condition that four points shall be in the same plane.*

1. Let OA, OB, OC, OD or α, β, γ, δ be the vectors to the four points; then $\delta - \alpha$, $\delta - \beta$, $\delta - \gamma$ are vectors in the same plane;

$$\therefore\ S\,.\,(\delta - \alpha)(\delta - \beta)(\delta - \gamma) = 0 \quad (31.\ 2.\ \text{Cor. } 2),$$

$$\text{or } S\,.\,\delta\beta\gamma + S\,.\,\alpha\delta\gamma + S\,.\,\alpha\beta\delta = S\,.\,\alpha\beta\gamma \quad (1).$$

2. Another form of the condition is to be obtained by assuming that

$$d\delta + c\gamma + b\beta + a\alpha = 0 \quad (2),$$

and substituting in equation (1) the value of δ deduced from this equation. The result is

$$\frac{a}{d} + \frac{b}{d} + \frac{c}{d} + 1 = 0,$$

$$\text{or } a + b + c + d = 0 \quad (3).$$

Equation (1), or the concurrence of equations (2) and (3) is the condition necessary and sufficient for coplanarity.

9. To find the line of intersection of two planes through the origin.

Let $S\alpha\rho = 0$, $S\beta\rho = 0$ be the planes.

Since every line in the one plane is perpendicular to α; and every line in the other perpendicular to β; the line required is perpendicular to both α and β, and is therefore parallel to $V\alpha\beta$, or $\rho = xV\alpha\beta$ is the equation.

10. The equation of the plane which passes through O and the line of intersection of the planes $S\alpha\rho = a$, $S\beta\rho = b$ is

$$S\rho\,(a\beta - b\alpha) = 0.$$

For 1° it is a plane through O; 2° if ρ be such that $S\alpha\rho = a$, then must $S\beta\rho = b$.

11. To find the equation of the line of intersection of two planes.

Let $$\rho = m\alpha + n\beta + xV\alpha\beta$$
be the equation required.

Then $$S\alpha\rho = m\alpha^2 + nS\alpha\beta,$$
since $V\alpha\beta$ is perpendicular to α, and similarly

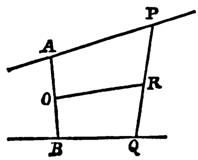
$$S\beta\rho = mS\alpha\beta + n\beta^2;$$

$$\therefore\; m = \frac{a\beta^2 - bS\alpha\beta}{\alpha^2\beta^2 - (S\alpha\beta)^2} = \frac{bS\alpha\beta - a\beta^2}{(V\alpha\beta)^2} \text{ (Art. 22. 9)},$$

$$n = \frac{aS\alpha\beta - b\alpha^2}{(S\alpha\beta)^2 - \alpha^2\beta^2} = \frac{aS\alpha\beta - b\alpha^2}{(V\alpha\beta)^2}.$$

35. We offer a few simple examples.

Ex. 1. *To find the locus of the middle points of all straight lines which are terminated by two given straight lines.*



Let AP, BQ be the two given straight lines, unit vectors parallel to which are β, γ; AB the line which is perpendicular to both AP, BQ.

Let O be the middle point of AB; vector $OA = \alpha$; R the middle point of any line PQ, vector $OR = \rho$; then

$$OP = \rho + RP = \alpha + x\beta,$$
$$OQ = \rho + RQ = -\alpha + y\gamma.$$

But
$$RP + RQ = 0;$$
$$\therefore\ 2\rho = x\beta + y\gamma;$$

hence, since
$$S\alpha\beta = 0,\ S\alpha\gamma = 0,$$

$S\alpha\rho = 0$ is the equation required; and the locus is a plane passing through O (33. Cor. 2), and perpendicular to OA (33. Cor. 1).

Note that, if $\beta \parallel \gamma$, we have simply

$$2\rho = x'\beta;$$

and, as there is now but one scalar indeterminate, the locus is a straight *line* instead of a *plane*.

Ex. 2. *Planes cut off, from the three rectangular co-ordinate axes, pyramids of equal volume, to find the locus of the feet of perpendiculars on them from the origin.*

Here the axes are *given*, so that i, j, k are known unit vectors.

Let ai, bj, ck be the portions cut off from the axes by a plane, the perpendicular on which from the origin is ρ.

Then $\rho - ai$ is perpendicular to ρ;

$$\therefore\ S\rho(\rho - ai) = 0,$$
$$\text{or}\ \rho^2 = aSi\rho.$$

Similarly,
$$\rho^2 = bSj\rho,$$
$$\rho^2 = cSk\rho.$$

Hence
$$\rho^6 = abc\, Si\rho\, Sj\rho\, Sk\rho$$
$$= C\, Si\rho\, Sj\rho\, Sk\rho,$$

since abc is by the problem constant.

If x, y, z be the co-ordinates of ρ this equation gives at once

$$(x^2 + y^2 + z^2)^3 = Cxyz$$

as the equation required.

Ex. 3. *To find the locus of the middle points of straight lines terminated by two given straight lines and all parallel to a given plane.*

Retaining the figure and notation of Ex. 1, let δ be the vector perpendicular to the given plane: we have

$$2\rho = x\beta + y\gamma,$$

$$2QP = 2\alpha + x\beta - y\gamma.$$

Now $$S\delta QP = 0 \quad (33.\ \text{Cor. } 3);$$

$$\therefore\ S\delta\,(2\alpha + x\beta - y\gamma) = 0;$$

$$\therefore\ y = \frac{2S\alpha\delta}{S\gamma\delta} + x\frac{S\beta\delta}{S\gamma\delta},$$

and $$2\rho = x\beta + \frac{2S\alpha\delta}{S\gamma\delta}\gamma + x\frac{S\beta\delta}{S\gamma\delta}\gamma$$

$$= a\gamma + x\,(\beta + b\gamma),$$

where $a = \frac{2S\alpha\delta}{S\gamma\delta}$, $b = \frac{S\beta\delta}{S\gamma\delta}$ are constants; ($S\gamma\delta$ for instance is the negative of the cosine of the angle between one of the given lines and the perpendicular to the given plane).

Now $\beta + b\gamma$ is a known vector lying between β and γ; call it ϵ, and $2\rho = a\gamma + x\epsilon$ is the equation required; which is that of a straight line, not generally passing through O (32. 1).

Ex. 4. *OA, OB are two fixed lines, which are cut by lines AB, $A'B'$ so that the area AOB is constant; and also the product OA, OA' constant. It is required to find the locus of the intersections of AB, $A'B'$.*

Let the unit vectors along OA, OB be α, β respectively.

$$OA = m\alpha,\quad OA' = m'\alpha,$$

$$OB = n\beta,\quad OB' = n'\beta;$$

then the conditions of the problem are

$$mn = m'n' = C,$$

$$mm' = a.$$

Now if AB, $A'B'$ intersect in P, and $OP=\rho$, we have

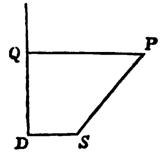
$$\rho = OA + AP$$
$$= m\alpha + x(n\beta - m\alpha),$$
$$\rho = OA' + A'P$$
$$= m'\alpha + x'(n'\beta - m'\alpha);$$

or $$\rho = m\alpha + x\left(\frac{C}{m}\beta - m\alpha\right),$$
$$\rho = m'\alpha + x'\left(\frac{C}{m'}\beta - m'\alpha\right);$$
$$\therefore\ m - xm = m' - x'm',$$
$$\frac{x}{m} = \frac{x'}{m'},$$
$$x = \frac{m}{m+m'}$$
$$= \frac{m^2}{m^2+a},$$
$$1 - x = \frac{a}{m^2+a},$$

and $$\rho = \frac{m}{m^2+a}(a\alpha + C\beta),$$

and the locus required is a straight line, the diagonal of the parallelogram whose sides are $a\alpha$, $C\beta$.

Ex. 5. *To find the locus of a point such that the ratio of its distances from a given point and a given straight line is constant—all in one plane.*



Let S be the given point, DQ the given straight line, $SP = ePQ$ the given relation.

Let vector $SD=\alpha$, $SP=\rho$, $DQ=y\gamma$, γ being the unit vector along DQ,

$$PQ = x\alpha;$$

then $$T\rho = eT(PQ),$$

gives $\rho^2 = e^2 PQ^2$, where PQ is a vector,

$$= e^2 (x\alpha)^2$$

$$= e^2 x^2 \alpha^2.$$

But $$\rho + x\alpha = SQ = SD + DQ$$

$$= \alpha + y\gamma;$$

$$\therefore\ S\alpha\rho + x\alpha^2 = \alpha^2, \text{ for } S\alpha\gamma = 0;$$

and $$x^2\alpha^4 = (\alpha^2 - S\alpha\rho)^2;$$

hence $$\alpha^2\rho^2 = e^2 (\alpha^2 - S\alpha\rho)^2,$$

a surface of the second order, whose intersection with the plane $S.\alpha\gamma\rho = 0$ is the required locus.

Ex. 6. *The same problem when the points and line are not in the same plane.*

Retaining the same figure and notation, we see that PQ is no longer a multiple of α; but

$$PQ = SQ - SP$$

$$= \alpha + y\gamma - \rho;$$

$$\therefore\ \rho^2 = e^2 (\alpha + y\gamma - \rho)^2,$$

and because PQ is perpendicular to DQ

$$S\gamma (\alpha + y\gamma - \rho) = 0;$$

$$\therefore\ (y\gamma^2, \text{ i. e.})\ -y = S\gamma\rho,$$

and $$\rho^2 = e^2 (\alpha - \gamma S\gamma\rho - \rho)^2,$$

a surface of the second order.

COR. If $e = 1$, and the surface be cut by a plane perpendicular to DQ whose equation is $S\gamma\rho = c$, the equation of the section is

$$\alpha^2 + c^2 - 2S\alpha\rho = 0,$$

another plane, so that the section is a straight line.

Ex. 7. *To find the locus of the middle points of lines of given length terminated by each of two given straight lines.*

Retaining the figure and notation of Ex. 1, and calling RP c, we have

$$2\rho = x\beta + y\gamma \quad (1),$$

and $$2RP = RP - RQ = 2\alpha + x\beta - y\gamma \quad (2).$$

From equation (1) we have

$$S\alpha\rho = 0 \quad (22.\ 7),$$
$$2S\beta\rho = -x + yS\beta\gamma,$$

because β is a unit vector,

$$2S\gamma\rho = xS\beta\gamma - y.$$

The first of these three equations shews that ρ lies in a plane through O perpendicular to AB (33. Cor. 2).

The second and third equations give

$$x = \frac{2\,(S\beta\rho + S\beta\gamma\, S\gamma\rho)}{(S\beta\gamma)^2 - 1},$$
$$y = \frac{2\,(S\gamma\rho + S\beta\gamma\, S\beta\rho)}{(S\beta\gamma)^2 - 1}.$$

Now (2) gives, by squaring,

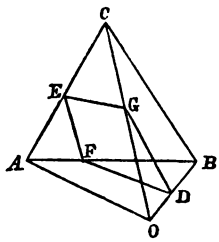
$$-4c^2 = 4\alpha^2 + x^2\beta^2 + y^2\gamma^2 - 2xyS\beta\gamma,$$

in which, if the values of x and y just obtained be substituted, there results an equation of the second order in ρ.

Hence the locus required is a plane curve of the second order, or a conic section, which by the very nature of the problem must be finite in extent and therefore an ellipse.

Ex. 8. *If a plane be drawn through the points of bisection of two opposite edges of a tetrahedron it will bisect the tetrahedron.*

Let D, E be the middle points of OB, AC: $DFEG$ the cutting plane: OA, OB, $OC = \alpha$, β, γ respectively.



$$OG = m\gamma,\ AF = n(\beta - \alpha).$$

The portion $ODGEA$ consists of three tetrahedra whose common vertex is O, and bases the triangles AEF, EFG, FGD.

Now $$OE = \frac{1}{2}(\gamma + \alpha),$$

$$OD = \frac{1}{2}\beta,$$

$$OG = m\gamma,$$

$$OF = \alpha + n(\beta - \alpha);$$

and 6 times the volume cut off

$$= S.\alpha\frac{1}{2}(\alpha+\gamma)\{\alpha+n(\beta-\alpha)\}$$

$$+ S.\frac{1}{2}(\alpha+\gamma)\,m\gamma\{\alpha+n(\beta-\alpha)\}$$

$$+ S.\{\alpha+n(\beta-\alpha)\}\,m\gamma\,\frac{1}{2}\beta \quad (31.\ 2.\ \text{Cor. } 1)$$

$$= \frac{1}{2}\{n+nm+(1-n)\,m\}\,S.\alpha\gamma\beta$$

$$= \frac{1}{2}(n+m)\,S.\alpha\gamma\beta.$$

But since E, G, D, F are in one plane, and

$$2m(1-n)\,OE - (1-n)\,OG + 2mnOD - mOF = 0,$$

we must have (34. 8)

$$2m(1-n) - (1-n) + 2mn - m = 0;$$

$$\therefore\ m + n = 1;$$

and 6 times the whole volume cut off

$$= \frac{1}{2}S.\alpha\gamma\beta$$

$$= \frac{1}{2} \text{ of 6 times the whole volume,}$$

hence the plane bisects the tetrahedron.

COR. The plane cuts other two edges at F and G, so that

$$\frac{AF}{AB} + \frac{OG}{OC} = 1.$$

ADDITIONAL EXAMPLES TO CHAP. IV.

1. Straight lines are drawn terminated by two given straight lines, to find the locus of a point in them whose distances from the extremities have a given ratio.

2. Two lines and a point S are given, not in one plane; find the locus of a point P such that a perpendicular from it on one of the given lines intersects the other, and the portion of the perpendicular between the point of section and P bears to SP a constant ratio. Prove that the locus of P is a surface of the second order.

3. Prove that the section of this surface by a plane perpendicular to the line to which the generating lines are drawn perpendicular is a circle.

4. Prove that the locus of a point whose distances from two given straight lines have a constant ratio is a surface of the second order.

5. A straight line moves parallel to a fixed plane and is terminated by two given straight lines not in one plane; find the locus of the point which divides the line into parts which have a constant ratio.

6. Required the locus of a point P such that the sum of the projections of OP on OA and OB is constant.

7. If the sum of the perpendiculars on two given planes from the point A is the same as the sum of the perpendiculars from B, this sum is the same for every point in the line AB.

8. If the sum of the perpendiculars on two given planes from each of three points A, B, C (not in the same straight line) be the same, this sum will remain the same for every point in the plane ABC.

9. A solid angle is contained by four plane angles. Through a given point in one of the edges to draw a plane so that the section shall be a parallelogram.

10. Through each of the edges of a tetrahedron a plane is drawn perpendicular to the opposite face. Prove that these planes pass through the same straight line.

11. *ABC* is a triangle formed by joining points in the rectangular co-ordinates *OA*, *OB*, *OC*; *OD* is perpendicular to *ABC*. Prove that the triangle *AOB* is a mean proportional between the triangles *ABC*, *ABD*.

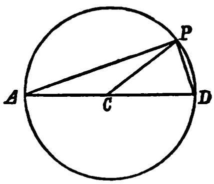
12. $Va\rho V\beta\rho + (Va\beta)^2 = 0$ is the equation of a hyperbola in ρ, the asymptotes being parallel to a, β.

CHAPTER V.

THE CIRCLE AND SPHERE.

36. *Equations of the circle.*

Let AD be the diameter of the circle, centre C, radius $= a$, P any point.



If vector $CD = \alpha$, $CP = \rho$,

we have $$\rho^2 = -a^2 \quad \text{............} (1).$$

If however $AP = \rho$,

$$CP = \rho - \alpha,$$

we have $$(\rho - \alpha)^2 = -a^2 \quad \text{............} (2).$$

If O be any point,

$$OP = \rho, \ OC = \gamma, \ CP = \rho - \gamma,$$

we have $$(\rho - \gamma)^2 = -a^2 \quad \text{........................} (3).$$

These are the three forms of the vector equation.

Form (2) may be written

$$\rho^2 - 2S\alpha\rho = 0.$$

If $OC = c$, form (3) may be written

$$\rho^2 - 2S\gamma\rho = c^2 - a^2.$$

Examples.

37. Ex. 1. *The angle in a semicircle is a right angle.*

Taking the second form

$$\rho^2 - 2S\alpha\rho = 0,$$

we may again write it

$$S\rho(\rho - 2\alpha) = 0;$$

therefore ρ, $\rho - 2\alpha$ are vectors at right angles to one another. But $\rho - 2\alpha$ is DP;

$$\therefore \ DPA \text{ is a right angle.}$$

Ex. 2. *If through any point O within or without a circle, a straight line be drawn cutting the circle in the points P, Q, the product OP . OQ is always the same for that point.*

The third form of the equation may be written

$$(T\rho)^2 + 2T\rho S\gamma U\rho + c^2 - a^2 = 0,$$

which shews that $T\rho$ has two values corresponding to each value of $U\rho$, the product of which is $c^2 - a^2$. Therefore, &c.

Ex. 3. *If two circles cut one another, the straight line which joins the points of section is perpendicular to the straight line which joins the centres.*

Let O, C be the centres, P, Q the points of section;

$$\text{vector } OC = \alpha;\ a,\ b \text{ the radii};$$

then (as vectors)

$$OP^2 = -a^2,$$

$$(OP - \alpha)^2 = -b^2;$$

$$\therefore \ S\alpha OP = C, \text{ a constant.}$$

Similarly, $$S\alpha OQ = C, \text{ the same constant};$$

$$\therefore \ S\alpha (OQ - OP) = 0,$$

$$\text{or } S\alpha PQ = 0,$$

$$\text{i. e. } PQ \text{ is at right angles to } OC.$$

Ex. 4. *O is a fixed point, AB a given straight line. A point Q is taken in the line OP drawn to a point P in AB, such that*

$$OP \,.\, OQ = k^2;$$

to find the locus of Q.

Let OA perpendicular to AB be a, vector α;

$$OQ = \rho,\ OP = x\rho;$$

then $$T(OP \,.\, OQ) = k^2,$$

or $$x\rho^2 = -k^2.$$

But $$Sa(x\rho - a) = 0;$$

$$\therefore\ xSa\rho = -a^2;$$

hence $$\rho^2 = \frac{k^2}{a^2} Sa\rho$$

is the equation of the locus of Q, which is therefore a circle, passing through O.

Ex. 5. *Straight lines are drawn through a fixed point, to find the locus of the feet of perpendiculars on them from another fixed point.*

Let O, A be the points, the lines being drawn through A. Let $OA = a$, and let $\rho = a + x\beta$ be the equation of one of the lines through A, δ the perpendicular on it from O.

Then $$\delta = a + x\beta,$$

and $$S\delta^2 = Sa\delta,$$

because δ is perpendicular to β;

$$\text{i. e. } \delta^2 - Sa\delta = 0,$$

the equation of a circle whose diameter is OA.

Ex. 6. *A chord QR is drawn parallel to the diameter AB of a circle: P is any point in AB; to prove that*

$$PQ^2 + PR^2 = PA^2 + PB^2.$$

Let $$CQ = \rho,\ CR = \rho',\ PC = a;$$

then $$PQ^2 = -(\text{vector } PQ)^2$$

$$= -(a+\rho)^2 = -(a^2 + 2Sa\rho + \rho^2),$$

$$PR^2 = -(a+\rho')^2 = -(a^2 + 2Sa\rho' + \rho'^2);$$

$$\therefore\ PQ^2 + PR^2 = 2PC^2 + 2AC^2 - 2(Sa\rho + Sa\rho').$$

But $$S(\rho+\rho')(\rho-\rho') = 0 \text{ and } \rho - \rho' = xa,$$

because QR is parallel to AB;

$$\therefore\ Sa\rho + Sa\rho' = 0,$$

and $$PQ^2 + PR^2 = 2PC^2 + 2AC^2$$

$$= PA^2 + PB^2.$$

Ex. 7. *If three given circles be cut by any other circle, the chords of section will form a triangle, the loci of the angular points of which are three straight lines respectively perpendicular to the lines which join the centres of the given circles; and these three lines meet in a point.*

Let A, B, C be the centres of the three given circles; a, b, c their radii; α, β, γ the vectors to A, B, C from the origin O; OA, OB, OC respectively p, q, r; D the centre of the cutting circle whose radius is R, $OD = s$, vector $OD = \delta$, ρ the vector to a point of section of circle D with circle A; then we shall have

$$(\rho - \alpha)^2 = -a^2, \quad (\rho - \delta)^2 = -R^2,$$

$$\text{and } \therefore\ 2S(\delta - \alpha)\rho = R^2 - a^2 - s^2 + p^2.$$

Now this is satisfied by the values of ρ to both points of section; and being the equation of a straight line (32. 3) is the equation of the line joining the points of section of circle D with circle A—call it line 1, and so of the others; then

$$\begin{aligned}
&\text{line 1 is } 2S(\delta - \alpha)\rho \ = R^2 - a^2 - s^2 + p^2,\\
&\text{line 2 is } 2S(\delta - \beta)\rho' = R^2 - b^2 - s^2 + q^2,\\
&\text{line 3 is } 2S(\delta - \gamma)\rho'' = R^2 - c^2 - s^2 + r^2.
\end{aligned}$$

If 1 and 2 intersect in P whose vector is ρ_1, 1 and 3 in Q (ρ_2); 2 and 3 in R (ρ_3), we shall have by subtraction

$$\begin{aligned}
&\text{at } P,\ 2S(\alpha - \beta)\rho_1 = a^2 - b^2 - p^2 + q^2;\\
&\text{at } Q,\ 2S(\gamma - \alpha)\rho_2 = -a^2 + c^2 + p^2 - r^2;\\
&\text{at } R,\ 2S(\beta - \gamma)\rho_3 = b^2 - c^2 - q^2 + r^2;
\end{aligned}$$

therefore (32. 3) the loci of P, Q, R are straight lines, perpendicular respectively to AB, AC, BC.

Also at the point of intersection of the first and third of these lines, we have, by addition,

$$2S(\alpha - \gamma)\rho = a^2 - c^2 - p^2 + r^2,$$

which is satisfied by the second: hence the three loci meet in a point.

Ex. 8. *To find the equation of the cissoid.*

AQ is a chord in a circle whose diameter is AB, QN perpendicular to AB.

AM is taken equal to BN, and MP is drawn perpendicular to AB to meet AQ in P; the locus of P is the cissoid.

Let vector $AP = \pi$, $AC = \alpha$, $AM = y\alpha$, $AQ = x\pi$;

then $y : 1 :: 2 - y : x$, by the construction;

$$\therefore\ y = \frac{2}{1+x}.$$

Now $$x^2\pi^2 - 2xS\alpha\pi = 0$$

is the equation of the circle;

$$\therefore\ x = \frac{2S\alpha\pi}{\pi^2}.$$

Also $$\pi = AM + MP$$

$$= y\alpha + \gamma;$$

$$\therefore\ S\alpha\pi = y\alpha^2,$$

$$y = \frac{S\alpha\pi}{\alpha^2};$$

hence $$\left(1 + \frac{2S\alpha\pi}{\pi^2}\right)\frac{S\alpha\pi}{\alpha^2} = 2,$$

and $$(\pi^2 + 2S\alpha\pi)\,S\alpha\pi = 2\alpha^2\pi^2,$$

is the equation required.

Ex. 9. *If $ABCD$ is a parallelogram, and if a circle be described passing through the point A, and cutting the sides AB, AC and the diagonal AD in the points F, G, H respectively; then the rectangle $AD.AH$ is equal to the sum of the rectangles $AB.AF$, and $AC.AG$.*

Let $$AB = \alpha,\ AC = \beta,\ AD = \gamma$$

$$= \alpha + \beta;$$

$$AF = x\alpha,\ AG = y\beta,\ AH = z\gamma;$$

θ the vector diameter of the circle; then

$$x\alpha^2 - S\alpha\theta = 0,$$
$$y\beta^2 - S\beta\theta = 0,$$
$$z\gamma^2 - S\gamma\theta = 0;$$

whence, since
$$\gamma = \alpha + \beta,$$
$$z\gamma^2 = x\alpha^2 + y\beta^2;$$
$$\text{i. e. } AD.AH = AB.AF + AC.AG.$$

Ex. 10. *What is represented by the equation*

$$\rho = (\alpha + x\beta)^{-1}?$$

If α, β be not at right angles to one another, we can put $\alpha_1 + e\beta$ for α, and so choose e that $S\alpha_1\beta = 0$.

We shall therefore consider α, β as vectors at right angles to each other, and we may, on account of x, assume their tensors equal, and each a unit.

Hence
$$\rho = \frac{\alpha + x\beta}{(\alpha + x\beta)^2} = -\frac{\alpha + x\beta}{1 + x^2},$$

or, if
$$\sin\theta = \frac{1}{\sqrt{1 + x^2}},$$
$$\cos\theta = \frac{x}{\sqrt{1 + x^2}},$$
$$\rho = -\sin\theta\,(\alpha\sin\theta + \beta\cos\theta),$$

whence
$$T\rho\,(=r) = \sin\theta,$$

a circle of which the diameter is a unit parallel to α and the origin a point in the circumference; and β a tangent vector at the origin.

Otherwise,
$$S\alpha\rho = \frac{1}{1 + x^2},$$
$$S\beta\rho = \frac{x}{1 + x^2};$$
$$\therefore\ (S\alpha\rho)^2 + (S\beta\rho)^2 = S\alpha\rho,$$
$$\text{or } -\rho^2 = S\alpha\rho.$$

Or, again, $$\rho^{-1} = \alpha + x\beta;$$
whence $$S\alpha\rho^{-1} = -1,$$
$$\text{or } V\beta(\rho^{-1} - \alpha) = 0,$$
$$\text{or } U\frac{\rho^{-1} - \alpha}{\beta} = 1,$$
where U stands for the versor of the quaternion;

all of these being, with the obvious condition $S.\alpha\beta\rho = 0$, varieties of the form of the equation of a circle, referred to a point in the circumference, the diameter through which is parallel to α.

Draw any two radii ρ and ρ_1, then we have

$$S.U\rho^{-1}U(\rho_1^{-1} - \rho^{-1}) = S.U\rho^{-1}U\frac{\rho_1\rho^2 - \rho_1^2\rho}{\rho_1^2\rho^2}$$
$$= S.U\rho^{-1}U\frac{\rho_1(\rho - \rho_1)\rho}{\rho_1^2\rho^2}.$$

Now $\frac{\rho_1(\rho - \rho_1)\rho}{\rho_1^2\rho^2}$ will be rendered a unit if we take a unit vector along each of the three vectors ρ_1, $(\rho - \rho_1)$, and ρ;

$$\therefore S.U\rho^{-1}U(\rho_1^{-1} - \rho^{-1}) = S.U\rho^{-1}U\rho_1U(\rho - \rho_1)U\rho$$
$$= S.U\rho_1U(\rho - \rho_1).$$

But $$\rho_1^{-1} - \rho^{-1} = (x_1 - x)\beta;$$
$$\therefore U(\rho_1^{-1} - \rho^{-1}) = \beta,$$
and $$S.U\rho^{-1}U(\rho_1^{-1} - \rho^{-1}) = S\beta U\rho^{-1} = -S\beta U\rho.$$
Hence $$S.U\rho_1U(\rho - \rho_1) = -S\beta U\rho.$$

If ρ be constant whilst ρ_1 varies, the right-hand side of this equation is constant, and the equation shews that the angles in the same segment of a circle are equal to one another.

Further, the form of the right-hand side of the equation, viz. $-S\beta U\rho$, shews that the angle in the segment is equal to the supplement of the angle between the chord (ρ) and the tangent (β).

38. *To draw a tangent to a circle.*

1. If we assume the first form of the equation, the centre being the origin, and assume also that the tangent is at right

angles to the radius drawn to the point of contact; we shall have, denoting by π a vector to a point in the tangent,

$$S\rho(\pi-\rho)=0,$$

for $\pi-\rho$ is along the tangent;

$$\therefore\ S\pi\rho=-a^2$$

is the equation required.

2. Without assuming the property of the tangent, we may obtain it as follows.

Let ρ' be a point in the circle near to P; then

$$S(\rho'^2-\rho^2)=0,$$

from the equation;

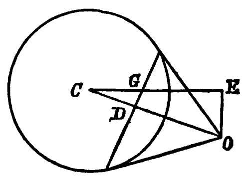
$$\text{i.e. } S(\rho'+\rho)(\rho'-\rho)=0.$$

But $\rho'+\rho$ is the vector which bisects the angle between the vectors to the points of section, and $\rho'-\rho$ is a vector along the secant.

Now the equation shews (22. 7) that the former of these lines is perpendicular to the latter.

As the points of section approach one another, the tangent approaches the secant, and the bisecting line approaches the radius to the point of contact: therefore the radius to the point of contact is perpendicular to the tangent.

39. From a point without a circle two tangents are drawn to the circle, to find the equation of the chord of contact.



Let β be the vector to the given point,

$$S\pi\rho=-a^2$$

the equation of a tangent; then since it passes through the given point

$$S\beta\rho=-a^2.$$

Now this equation is satisfied for both points of contact, and since it is the equation of a straight line (32. 3) it must be satisfied for every point in the straight line which passes through those points: it is therefore the equation of the chord of contact. To

avoid the appearance of limiting ρ to a point in the circle, we may write σ in place of ρ; and the equation of the chord of contact becomes

$$S\beta\sigma = -a^2.$$

EXAMPLES.

40. Ex. 1. *If chords be drawn through a given point, and tangents be drawn at the points of section, the corresponding pairs of tangents will intersect in a straight line.*

Let γ be the vector to the given point G, the centre C being the origin; β the vector to O, the point of intersection of two tangents at the extremities of a chord through G; then the equation of the chord of contact is (39)

$$S\beta\sigma = -a^2,$$

and as the chord passes through G we have

$$S\beta\gamma = -a^2,$$

which, since γ is a constant vector, is the equation of a straight line, the locus of β.

COR. 1. The straight line is at right angles to CG (32. 3).

COR. 2. The converse is obviously true, that if through points in a straight line pairs of tangents be drawn to a circle, the chords of contact all pass through the same point.

Ex. 2. *Any chord drawn from O the point of intersection of two tangents, is cut harmonically by the circle and the chord of contact.*

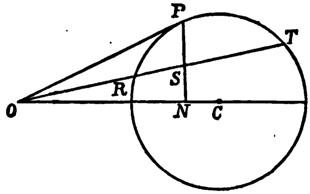
Let radius $= a$, $OC = c$, $OR = p$, $OS = q$, vector $OC = \alpha$, unit vector $OR = \rho$; then

$$(p\rho)^2 - 2pS\alpha\rho = c^2 - a^2$$

is the equation of the circle;

$$\text{i. e. } p^2 + 2pS\alpha\rho + c^2 - a^2 = 0,$$

a quadratic equation which gives the two values of p, viz. OR and OT;



$$\therefore \frac{1}{OR}+\frac{1}{OT}=-\frac{2S\alpha\rho}{c^2-a^2}.$$

But $q\rho = OS = ON + NS$,

$$Saq\rho = S\alpha ON;$$

i. e. $qS\alpha\rho = S\alpha(OC - NC)$,

$$= a^2 - S\alpha NC$$

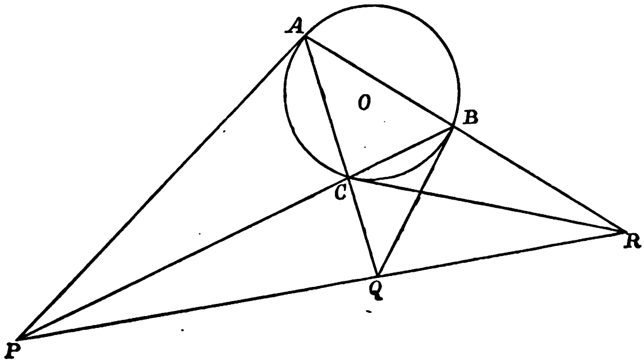
$$= -c^2 + a^2 \text{ (39)};$$

hence
$$\frac{2}{OS}=\frac{2}{q}$$

$$= -\frac{2S\alpha\rho}{c^2-a^2}$$

$$=\frac{1}{OR}+\frac{1}{OT}.$$

Ex. 3. *If tangents be drawn at the angular points of a triangle inscribed in a circle, the intersections of these tangents with the opposite sides of the triangle lie in a straight line.*



Let radius $= a$, $OA = \alpha$, $OB = \beta$, $OC = \gamma$, then

$$OP = \alpha + xAP = \beta + y(\gamma - \beta).$$

But a is perpendicular to AP;

$$\therefore \; Sa^2 = Sa\beta + yS(a\gamma - a\beta),$$

$$y = -\frac{a^2 + Sa\beta}{Sa\gamma - Sa\beta},$$

and
$$OP = \frac{(a^2 + Sa\gamma)\beta - (a^2 + Sa\beta)\gamma}{Sa\gamma - Sa\beta}.$$

Similarly,
$$OQ = \frac{(a^2 + Sa\beta)\gamma - (a^2 + S\beta\gamma)a}{Sa\beta - S\beta\gamma},$$

$$OR = \frac{(a^2 + S\beta\gamma)a - (a^2 + Sa\gamma)\beta}{S\beta\gamma - Sa\gamma}.$$

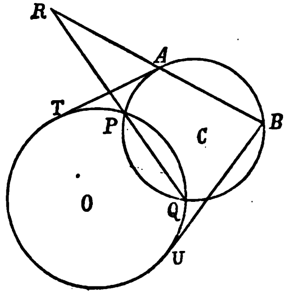
Hence
$$(Sa\gamma - Sa\beta)\,OP + (Sa\beta - S\beta\gamma)\,OQ + (S\beta\gamma - Sa\gamma)\,OR = 0,$$

whilst
$$(Sa\gamma - Sa\beta) + (Sa\beta - S\beta\gamma) + (S\beta\gamma - Sa\gamma) = 0.$$

Consequently (Art. 13) P, Q, R are in the same straight line.

COR.
$$\begin{aligned} PQ : PR &:: S\beta\gamma - Sa\gamma :: S\beta\gamma - Sa\beta \\ &:: \cos 2B - \cos 2A : \cos 2C - \cos 2A \\ &:: \sin C \sin(B - A) : \sin B \sin(C - A). \end{aligned}$$

Ex. 4. *A fixed circle is cut by a number of circles, all of which pass through two given points; to prove that the lines of section of the fixed circle with each circle of the series all pass through a point whose distances from the two given points are proportional to the squares of the tangents drawn from those points to the fixed circle.*



Let O be the centre of the fixed circle whose radius is a, A, B the given points, vectors a, β, the origin being O; $OA = b$, $OB = c$; C the centre of a circle which passes through A and B, radius r; $OC = \rho$, π the vector to any point in the circumference of this circle; then the equation of the circle is $(\pi - \rho)^2 = -r^2$;

hence for the four points A, B, P, Q, we have

$$\alpha^2 - 2S\alpha\rho + \rho^2 = -r^2,$$
$$\beta^2 - 2S\beta\rho + \rho^2 = -r^2,$$
$$OP^2 - 2S\,.\,OP\rho + \rho^2 = -r^2,$$
$$OQ^2 - 2S\,.\,OQ\rho + \rho^2 = -r^2.$$

From which it follows that

$$S(OP - OQ)\,\rho = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$
$$-b^2 + c^2 = \alpha^2 - \beta^2 = 2S(\alpha - \beta)\,\rho \ldots\ldots\ldots\ldots\ldots (2),$$
$$2S(OP - \alpha)\,\rho = OP^2 - \alpha^2 = -a^2 + b^2 \ldots\ldots\ldots\ldots (3).$$

Let QP, AB intersect in R, $OR = \sigma$; then

$$S\sigma\rho = S\{OP + x(OP - OQ)\}\,\rho$$
$$= S\,.\,OP\rho \text{ by (1)},$$

and

$$S\sigma\rho = S\{\alpha + y(\alpha - \beta)\}\,\rho$$
$$= S\alpha\rho + \frac{y}{2}(-b^2 + c^2) \text{ by (2)};$$

$$\therefore\ y(-b^2 + c^2) = 2S\sigma\rho - 2S\alpha\rho$$
$$= 2S(OP - \alpha)\,\rho$$
$$= -a^2 + b^2 \text{ by (3)},$$

i.e. y is independent of ρ and r; or R is the same point for every circle:

also
$$OR = \frac{(c^2 - a^2)\,\alpha - (b^2 - a^2)\,\beta}{c^2 - b^2},$$

and
$$RA : RB :: \alpha - OR : \beta - OR :: b^2 - a^2 : c^2 - a^2$$
$$:: AT^2 : BU^2.$$

41. *The Sphere.*

1. It is clear that there is nothing in the demonstration of Art. 36 which limits the conclusions to one plane; it follows that the equations there obtained are also equations of a sphere.

2. Further if we assume that the tangent plane to a sphere is perpendicular to the radius to the point of contact, the conclusion in Art. 38 is applicable also.

The equation of the tangent plane to the sphere is therefore

$$S\pi\rho = -a^2.$$

3. Lastly, the results of Art. 39 are also applicable if we substitute any number of tangent planes passing through a given point for two tangent lines; the equation of the plane which passes through the points of contact is therefore

$$S\beta\sigma = -a^2.$$

This plane is the *polar* plane to the point through which the tangent planes pass.

COR. Since the polar plane is perpendicular to the line which joins the centre with the point through which the tangent planes pass, the perpendicular CD to it from the centre is along this line and has therefore the same unit vector with it. The equation above gives in this case

$$S\{CO \,.\, CD\,(U\beta)^2\} = -a^2;$$
$$\therefore\ CO \,.\, CD = a^2 \quad (19).$$

EXAMPLES.

42. Ex. 1. *Every section of a sphere made by a plane is a circle.*

Let $\rho^2 = -a^2$ be the equation of the sphere, α the vector perpendicular from the centre on the cutting plane; c the corresponding line.

Let $$\rho = \alpha + \pi;$$
then the equation becomes

$$\pi^2 + 2S\alpha\pi + \alpha^2 = -a^2.$$

But $$S\alpha\pi = 0;$$

$$\therefore\ \pi^2 = -(a^2 - c^2)$$

is the equation of the section, which is therefore a circle, the square of whose radius is $a^2 - c^2$.

Ex. 2. *To find the curve of intersection of two spheres.*

Let the equations be

$$\rho^2 - 2S\alpha\rho = C,$$
$$\rho^2 - 2S\alpha'\rho = C';$$

$$\therefore\ 2S(\alpha' - \alpha)\rho = C - C',$$

a plane perpendicular to the line of which the vector is $\alpha' - \alpha$, i. e. to the line which joins the centres of the two circles.

Hence, by Ex. 1, the curve of intersection is a circle.

Ex. 3. *To find the locus of the feet of perpendiculars from the origin on planes which pass through a given point.*

Let α be the vector to the point, δ perpendicular on a plane through it; then

$$S\delta(\rho - \alpha) = 0$$

is the equation of that plane; therefore for the foot of the perpendicular

$$S(\delta^2 - \alpha\delta) = 0;$$

or
$$\delta^2 - S\alpha\delta = 0$$

is true for the foot of every perpendicular and is therefore the equation of the surface required. Hence it is a sphere whose diameter is the line joining the origin with the given point.

Ex. 4. *Perpendiculars are drawn from a point on the surface of a sphere to all tangent planes, to find the locus of their extremities.*

Let α be the vector to the given point,

$$S\pi\rho = -a^2$$

the equation of a tangent plane.

Since the perpendicular is parallel to ρ, its vector is

$$\pi = \alpha + x\rho;$$

$$\therefore\ (\pi - \alpha)^2 = x^2\rho^2 = x^2\alpha^2$$
$$= -x^2a^2;$$

because both ρ and α are vector radii.

But $S\pi\rho = -a^2$ gives with $x\rho = \pi - \alpha$,

$$S\pi(\pi - \alpha) = -a^2x,$$
$$(\pi^2 - S\alpha\pi)^2 = a^4x^2$$
$$= -a^2 \times -a^2x^2$$
$$= -a^2(\pi - \alpha)^2.$$

Ex. 5. *If the points from which tangent planes are drawn to a sphere lie always in a straight line, prove that the planes of section all pass through a given point.*

Let CE be perpendicular to the line in which the point β lies (41), see fig. of Art. 39,

$$CE = c, \text{ vector } CE = \delta;$$

then

$$S\beta\delta = -c^2$$

is the equation of the line.

But

$$S\beta\sigma = -a^2$$

is the plane of contact, which is therefore satisfied by

$$\sigma = \frac{a^2}{c^2}\delta,$$

i. e. the planes all pass through a point G in CE, such that

$$CG = \frac{a^2}{c^2} CE,$$

$$\text{or } CE \cdot CG = a^2.$$

Ex. 6. *If three spheres intersect one another, their three planes of intersection all pass through the same straight line.*

Let α, β, γ be the vectors to the centres of the three spheres,

$$\rho^2 - 2S\alpha\rho = a,$$
$$\rho^2 - 2S\beta\rho = b,$$
$$\rho^2 - 2S\gamma\rho = c,$$

their three equations;

$$\therefore\ 2S(\alpha - \beta)\rho = b - a,$$
$$2S(\alpha - \gamma)\rho = c - a,$$
$$2S(\beta - \gamma)\rho = c - b,$$

are the equations of the three planes of intersection.

Now the line of intersection of the first and second of these planes is obtained by taking ρ so as to satisfy both equations, and therefore their difference

$$2S(\beta - \gamma)\rho = c - b,$$

which, being the third equation, proves that the same value of ρ satisfies it also. The three planes consequently all pass through the same straight line.

Ex. 7. *To find the locus of a point, the sum of the squares of whose distances from a number of given points has a given value.*

Let ρ denote the sought point; α, β, ... the given ones; then

$$(\rho-\alpha)^2+(\rho-\beta)^2+\&c.=\Sigma(\rho-\alpha)^2=-C.$$

If there be n given points; this is

$$n\rho^2-2S.\rho\Sigma\alpha+\Sigma\alpha^2=-C,$$

or

$$\left(\rho-\frac{\Sigma\alpha}{n}\right)^2=\left(\frac{\Sigma\alpha}{n}\right)^2-\frac{1}{n}(\Sigma.\alpha^2+C).$$

This is the equation of a sphere, the vector to whose centre is

$$\frac{1}{n}\Sigma(\alpha),$$

i. e. the centre of inertia of the n points taken as equal.

Transpose the origin to this point, then (36)

$$\Sigma.\alpha=0,$$

and

$$\rho^2=-\frac{1}{n}\{\Sigma(\alpha^2)+C\}.$$

Hence, that there may be a real locus, C must be positive and not less than the sum of the squares of the distances of the given system of points from their centre of inertia. If C have its least value, we have of course

$$\rho^2=0,$$

the sphere having shrunk to a point.

ADDITIONAL EXAMPLES TO CHAP. V.

1. If two circles cut one another, and from one of the points of section diameters be drawn to both circles, their other extremities and the other point of section will be in a straight line.

2. If a chord be drawn parallel to the diameter of a circle, the radii to the points where it meets the circle make equal angles with the diameter.

3. The locus of a point from which two unequal circles subtend equal angles is a circle.

4. A line moves so that the sum of the perpendiculars on it from two given points in its plane is constant. Shew that the locus of the middle point between the feet of the perpendiculars is a circle.

5. If O, O' be the centres of two circles, the circumference of the latter of which passes through O; then the point of intersection A of the circles being joined with O' and produced to meet the circles in C, D, we shall have

$$AC \,.\, AD = 2AO'^2.$$

6. If two circles touch one another in O, and two common chords be drawn through O at right angles to one another, the sum of their squares is equal to the square of the sum of the diameters of the circles.

7. A, B, C are three points in the circumference of a circle; prove that if tangents at B and C meet in D, those at C and A in E, and those at A and B in F; then AD, BE, CF will meet in a point.

8. If A, B, C are three points in the circumference of a circle, prove that $V(AB \,.\, BC \,.\, CA)$ is a vector parallel to the tangent at A.

9. A straight line is drawn from a given point O to a point P on a given sphere: a point Q is taken in OP so that

$$OP \,.\, OQ = k^2.$$

Prove that the locus of Q is a sphere.

10. A point moves so that the ratio of its distances from two given points is constant. Prove that its locus is either a plane or a sphere.

11. A point moves so that the sum of the squares of its distances from a number of given points is constant. Prove that its locus is a sphere.

12. A sphere touches each of two given straight lines which do not meet; find the locus of its centre.

CHAPTER VI.

THE ELLIPSE.

43. 1. If we define a conic section as "the locus of a point which moves so that its distance from a fixed point bears a constant ratio to its distance from a fixed straight line" (Todhunter, Art. 123), we shall find the equation to be (Ex. 5, Art. 35)

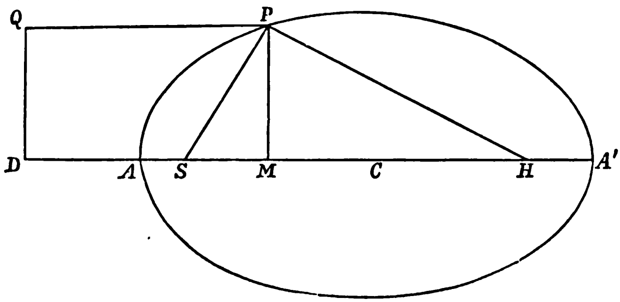
$$\alpha^2\rho^2 = e^2(\alpha^2 - S\alpha\rho)^2 \text{.......................}(1),$$

where $\qquad SP = ePQ$, vector $SD = \alpha$, $SP = \rho$.

When e is less than 1, the curve is the ellipse, a few of whose properties we are about to exhibit.

2. SA, SA' are multiples of α: call one of them $x\alpha$: then, by equation (1), putting $x\alpha$ for ρ, we get

$$x^2 = e^2(1-x)^2;$$



$$\therefore\ x = \frac{e}{1+e},$$

$$x = -\frac{e}{1-e},$$

$$\text{i.e. } SA = \frac{e}{1+e} SD,$$

$$SA' = \frac{e}{1-e} SD;$$

$$\therefore\ AA' = \frac{2e}{1-e^2} SD,$$

the major axis of the ellipse, which we shall as usual abbreviate by $2a$.

If C be the centre of the ellipse

$$CS = SA' - CA' = \left(\frac{e}{1-e} - \frac{e}{1-e^2}\right) SD = eCA$$
$$= ae,$$

and if vector CS be designated by α', CP by ρ', we have

$$\alpha' = \frac{e^2}{1-e^2}\alpha \text{ and } \rho' = \rho + \alpha';$$

whence, by substituting in (1), the equation assumes the form

$$a^2\rho'^2 + (S\alpha'\rho')^2 = -a^4(1-e^2);$$

which we may now write, CS being α and CP ρ,

$$a^2\rho^2 + (S\alpha\rho)^2 = -a^4(1-e^2) \quad\text{..................(2).}$$

3. This equation might have been obtained at once by referring the ellipse to the two foci, as Newton does in the Principia, Book I. Prop. 11; the definition then becomes

$$SP + HP = 2a,$$

or in vectors, if

$$CP = \rho,\ CS = \alpha,$$

$$T(\rho + \alpha) + T(\rho - \alpha) = 2a;$$

$$\text{i.e. } \sqrt{-(\rho+\alpha)^2} + \sqrt{-(\rho-\alpha)^2} = 2a;$$

hence, squaring,

$$a\sqrt{-(\rho-\alpha)^2} = a^2 + S\alpha\rho;$$

$$\text{i.e. } a^2\rho^2 + (S\alpha\rho)^2 = -a^4(1-e^2).$$

If now we write $\phi\rho$ for $-\dfrac{a^2\rho + \alpha S\alpha\rho}{a^4(1-e^2)}$, where $\phi\rho$ is a vector which coincides with ρ only in the cases in which either α coincides with ρ or when $S\alpha\rho = 0$, i. e. in the cases of the principal axes; the equation of the ellipse becomes

$$S\rho\phi\rho = 1 \text{..............................(3).}$$

The same equation is, of course, applicable to the hyperbola, e being greater than 1.

44. The following properties of $\phi\rho$ will be *very frequently* employed. The reader is requested to bear them constantly in mind.

1. $\phi(\rho + \sigma) = \phi\rho + \phi\sigma.$

2. $\phi x\rho = x\phi\rho.$

3. $S\sigma\phi\rho = -\dfrac{a^2 S\sigma\rho + S\alpha\sigma S\alpha\rho}{a^4(1-e^2)}$

$$= S\rho\phi\sigma.$$

They need no other demonstration than what results from simple inspection of the value of $\phi\rho$

$$= -\frac{a^2\rho + \alpha S\alpha\rho}{a^4(1-e^2)}.$$

45. To find the equation of the tangent to the ellipse.

The tangent is defined to be the limit to which the secant approaches as the points of section approach each other.

Let $CP = \rho$, $CQ = \rho'$, then

$$\text{vector } PQ = CQ - CP = \rho' - \rho = \beta \text{ say;}$$

β is therefore a vector along the secant.

Now
$$\begin{aligned} S\rho'\phi\rho' &= S(\rho + \beta)\phi(\rho + \beta) \\ &= S(\rho + \beta)(\phi\rho + \phi\beta) \quad (44.\ 1) \\ &= S\rho\phi\rho + S\rho\phi\beta + S\beta\phi\rho + S\beta\phi\beta. \end{aligned}$$

But $$S\rho'\phi\rho' = 1 = S\rho\phi\rho;$$

$$\therefore\ S\rho\phi\beta + S\beta\phi\rho + S\beta\phi\beta = 0,$$

or (44. 3) $$2S\beta\phi\rho + S\beta\phi\beta = 0.$$

Now $\beta\phi\rho$ involves the first power of β whilst $\beta\phi\beta$ involves the second, and the definition requires that the limit of the sum of the two as β gets smaller and smaller should be the first only, even if that should be zero: i. e. when β is along the tangent, we must have

$$2S\beta\phi\rho = 0.$$

Let then T be any point in the tangent, $CT = \pi$, then

$$\pi = \rho + x\beta,$$

and $S\beta\phi\rho = 0$ gives

$$S(\pi - \rho)\phi\rho = 0;$$

$$\therefore\ S\pi\phi\rho = S\rho\phi\rho = 1$$

is the equation of the tangent.

COR. 1. $\phi\rho$ is a vector along the perpendicular to the tangent (32. 3), that is, $\phi\rho$ is a normal vector, or parallel to a normal vector at the point ρ.

COR. 2. The equation of the tangent may also be written (44. 3) $S\rho\phi\pi = 1$.

46. We may now exhibit the corresponding equations in terms of the Cartesian co-ordinates, as some of the results are best known in that form.

Let $CM = x$, $MP = y$ as usual; then, retaining the notation of Art. 31 with i, j as unit vectors parallel and perpendicular respectively to CA,

$$\text{vector } CM = xi,\ MP = yj,\ CS = aei;$$

$$\therefore\ \rho = xi + yj,$$

$$\phi\rho = -\frac{a^2\rho + \alpha S\alpha\rho}{a^4(1-e^2)}$$

$$= -\frac{a^2(1-e^2)xi + a^2yj}{a^4(1-e^2)}$$

$$= -\left(\frac{xi}{a^2} + \frac{yj}{b^2}\right),$$

where $$b^2 = a^2(1-e^2);$$

and $$S\rho\phi\rho = -S(xi + yj)\left(\frac{xi}{a^2} + \frac{yj}{b^2}\right)$$
$$= \frac{x^2}{a^2} + \frac{y^2}{b^2};$$
$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is the Cartesian interpretation of $S\rho\phi\rho = 1$.

Again, if x', y' be the co-ordinates of T a point in the tangent,
$$\pi = x'i + y'j,$$

and $$S\pi\phi\rho = -S(x'i + y'j)\left(\frac{xi}{a^2} + \frac{yj}{b^2}\right)$$
$$= \frac{xx'}{a^2} + \frac{yy'}{b^2};$$
$$\therefore \frac{xx'}{a^2} + \frac{yy'}{b^2} = 1$$

is the equation of the tangent.

47. The values of ρ and $\phi\rho$ exhibited in the last Article, viz.
$$\rho = xi + yj,\ \phi\rho = -\left(\frac{xi}{a^2} + \frac{yj}{b^2}\right) \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

enable us to write
$$\phi\rho = \frac{iSi\rho}{a^2} + \frac{jSj\rho}{b^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

We shall have
$$\phi^2\rho = \phi\phi\rho = \frac{iSi\phi\rho}{a^2} + \frac{jSj\phi\rho}{b^2}$$
$$= -\left(\frac{iSi\rho}{a^4} + \frac{jSj\rho}{b^4}\right) \ldots\ldots\ldots\ldots\ldots\ldots(3),$$
$$\phi^{-1}\rho = a^2 iSi\rho + b^2 jSj\rho,\ \&c.$$

If, further, we write $\psi\rho$ for
$$-\left(\frac{iSi\rho}{a} + \frac{jSj\rho}{b}\right),$$

we shall have

$$\psi^2\rho = \psi\psi\rho = -\left(\frac{iSi\rho}{a^2} + \frac{jSj\rho}{b^2}\right)$$

$$= -\phi\rho \quad \text{.................................} (4),$$

$$\psi^{-1}\rho = -(aiSi\rho + bjSj\rho), \text{ \&c.}$$

$$\rho = \psi^{-1}\psi\rho$$

$$= -(aiSi\psi\rho + bjSj\psi\rho) \quad \text{..................} (5).$$

It is evident that the properties of $\phi\rho$ (Art. 44) are applicable to all these functions.

Now $$S\rho\phi\rho = 1$$

gives $$S\rho\psi(\psi\rho) = -1.$$

But since $$S\rho\psi\sigma = S\sigma\psi\rho,$$

this becomes $$S\psi\rho\psi\rho = -1,$$

or $$T\psi\rho = 1;$$

which shews 1. that $\psi\rho$ is a unit vector; 2. that the equation of the ellipse may be expressed in the *form* of the equation of a circle, the vector which represents the radius being itself of variable length, deformed by the function ψ.

Lastly, $$S\alpha\phi\beta = 0$$

gives $$S\alpha\psi^2\beta = S\psi\alpha\psi\beta = 0;$$

therefore $\psi\alpha$, $\psi\beta$ are vectors at right angles to one another.

48. To find the locus of the middle points of parallel chords.

Let all the chords be parallel to the vector β; π the vector to the middle point of one of them whose vector length is $2x\beta$; then

$$\pi + x\beta, \quad \pi - x\beta$$

are vectors to points in the ellipse;

$$\therefore\ S(\pi + x\beta)\,\phi\,(\pi + x\beta) = 1,$$

$$S(\pi - x\beta)\,\phi\,(\pi - x\beta) = 1,$$

multiplying out, observing that (44. 1),

$$\phi(\pi + x\beta) = \phi\pi + x\phi\beta, \text{ \&c.},$$

we get by subtracting,

$$S\pi\phi\beta + S\beta\phi\pi = 0,$$

or, (Art. 44. 3),

$$2S\pi\phi\beta = 0;$$

$$\therefore\ S\pi\phi\beta = 0,$$

i.e. the locus required is a straight line perpendicular to $\phi\beta$.

Now $\phi\beta$ is the vector perpendicular to the tangent at the extremity of the diameter β (Art. 45. Cor. 1).

Therefore the locus of the middle points of parallel chords is a diameter parallel to the tangent at the extremity of the diameter to which the chords are parallel.

COR. If α be the diameter which bisects all chords parallel to β; since

$$S\alpha\phi\beta = 0,$$

we have (Art. 44. 3),

$$S\beta\phi\alpha = 0,$$

which is the equation to the straight line that bisects all chords parallel to α. Moreover β is parallel to the tangent at the extremity of α, for it is perpendicular to the normal $\phi\alpha$.

Hence the properties of α with respect to β are convertible with those of β with respect to α: and the diameters which satisfy the equation

$$S\alpha\phi\beta = 0,$$

are said to be conjugate to one another.

49. Our object being simply to illustrate the process, we shall set down in this Article a few of the properties of conjugate diameters without attempting to classify or complete them.

1. If CP, CD are the conjugate semi-diameters α, β; and if DC be produced to meet the ellipse again in E, and PD, PE be joined; vector $DP = \alpha - \beta$, vector $EP = \alpha + \beta$.

Now

$$S(\alpha+\beta)\,\phi(\alpha-\beta)=S(\alpha+\beta)\,(\phi\alpha-\phi\beta)$$
$$=S\alpha\phi\alpha-S\beta\phi\beta-S\alpha\phi\beta+S\beta\phi\alpha \quad (44.\ 1)$$
$$=0,$$

because $S\alpha\phi\alpha$, $S\beta\phi\beta$, each equals 1.

Therefore $\alpha+\beta$, $\alpha-\beta$ are parallel to conjugate diameters. (Art. 48. Cor.)

This is the property of *Supplemental Chords.*

2. Let two tangents meet in T, $CT=\pi$, and let the chord of contact be parallel to β. If for the present purpose we denote CN by α, we have

$$S\pi\phi\,(\alpha+x\,\beta)=1,$$
$$S\pi\phi\,(\alpha+x_1\,\beta)=1,$$

for the two points of contact.

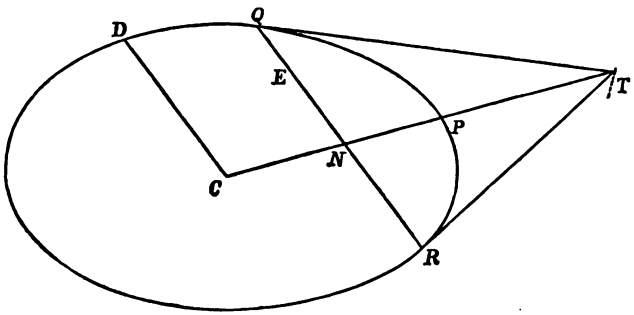
Subtracting and applying (44. 1),

$$S\pi\phi\beta=0:$$

hence π and β i.e. CT, QR are conjugate.

3. The equation of the chord of contact is $S\sigma\phi\pi=1$.

For $S\rho\phi\pi=1$ (45. Cor. 2) is satisfied by the values of ρ at Q and at R, and since $S\rho\phi\pi=1$ or $S\sigma\phi\pi=1$ is the equation



of a straight line, π being a constant vector (32. 3) it is the line QR.

4. If QR pass through a fixed point E, the locus of T is a straight line.

Let σ be the vector to the point E, then

$$S\sigma\phi\pi = 1;$$

$$\therefore\ S\pi\phi\sigma = 1,$$

or the locus of T is a straight line perpendicular to $\phi\sigma$, i.e. parallel to the tangent at the point where CE meets the ellipse. (45. Cor. 1.)

The converse is of course true.

5. Let us now take

$$CP = \alpha,\ \ CD = \beta,\ \ CN = x\alpha,\ \ NQ = y\beta,\ \ CT = z\alpha;$$

then the equation of the tangent becomes

$$Sz\alpha\phi\,(x\alpha + y\beta) = 1;$$

$$\text{i.e. } xzS\alpha\phi\alpha = 1;$$

$$\therefore\ xz = 1,$$

$$\text{or } x\alpha\,.\,z\alpha = \alpha^2;$$

geometrically $$CN\,.\,CT = CP^2.$$

6. The equation of the ellipse gives

$$S\,(x\alpha + y\beta)\,\phi\,(x\alpha + y\beta) = 1,$$

or $$x^2S\alpha\phi\alpha + y^2S\beta\phi\beta + 2xyS\alpha\phi\beta = 1,$$

$$\text{i.e. } x^2 + y^2 = 1,$$

or, since CN is $x\alpha$, $CP = \alpha$, &c.,

$$\left(\frac{CN}{CP}\right)^2 + \left(\frac{NQ}{CD}\right)^2 = 1,$$

the equation of the ellipse referred to conjugate diameters.

7. $$\alpha = \psi^{-1}\psi\alpha = -\,(aiSi\psi\alpha + bjSj\psi\alpha)$$

$$\beta = \psi^{-1}\psi\beta = -\,(aiSi\psi\beta + bjSj\psi\beta);$$

$$\therefore\ V\alpha\beta = ab\,Vij\,(Si\psi\alpha Sj\psi\beta - Si\psi\beta Sj\psi\alpha).$$

If now we call k the unit vector perpendicular to the plane of the ellipse, we get

$$Vij = k.$$

And, observing that $\psi\alpha$, $\psi\beta$ are unit vectors at right angles; if the angle between i and $\psi\alpha$ be θ, that between i and $\psi\beta$ will be $\frac{\pi}{2} + \theta$, &c. &c., we shall have (21. 3)

$$Si\psi\alpha = -\cos\theta,$$

$$Si\psi\beta = \sin\theta,$$

$$Sj\psi\alpha = -\sin\theta,$$

$$Sj\psi\beta = -\cos\theta.$$

$$\therefore\ Si\psi\alpha Sj\psi\beta - Si\psi\beta Sj\psi\alpha = \cos^2\theta + \sin^2\theta = 1.$$

Consequently $$V\alpha\beta = abk;$$

i. e. $$T\alpha\,.\,T\beta \sin PCD = ab,$$

or *all parallelograms circumscribing an ellipse are equal.*

50. Examples.

Ex. 1. *To find the length of the perpendicular from the centre on the tangent.*

Let CY the perpendicular, which (Art. 45. Cor. 1) is a vector along $\phi\rho$, be $x\phi\rho$; then since Y is a point in the tangent,

$$S\varpi\phi\rho = 1 \text{ gives } Sx\phi\rho\phi\rho = 1,$$

$$\text{or } x(\phi\rho)^2 = 1;$$

$$\therefore\ (x\phi\rho)^2(\phi\rho)^2 = 1,$$

and $$CY^2 = T(x\phi\rho)^2 = T\frac{1}{(\phi\rho)^2}$$

$$= \frac{1}{\frac{x^2}{a^4} + \frac{y^2}{b^4}}\ (46).$$

Ex. 2. *The product of the perpendiculars from the foci on the tangent is equal to the square of the semi-axis minor.*

We have SY the vector perpendicular $= x\phi\rho$, and as Y is a point in the tangent, and

$$CY = \alpha + x\phi\rho,$$

$$S(\alpha + x\phi\rho)\,\phi\rho = 1,$$

$$x(\phi\rho)^2 = 1 - S\alpha\phi\rho,$$

$$SY = Tx\phi\rho = T\frac{1 - S\alpha\phi\rho}{\phi\rho}.$$

Similarly,
$$HZ = T\frac{1 + S\alpha\phi\rho}{\phi\rho};$$

$$\therefore\ SY \,.\, HZ = T\frac{1 - S^2\alpha\phi\rho}{(\phi\rho)^2}.$$

Now (43. 2)
$$a^2\rho^2 = -S^2\alpha\rho - a^4(1 - e^2),$$

$$\phi\rho = -\frac{a^2\rho + \alpha S\alpha\rho}{a^4(1 - e^2)};$$

$$\therefore\ (\phi\rho)^2 = \frac{S^2\alpha\rho - a^4}{a^6(1 - e^2)},$$

$$1 - S^2(\alpha\phi\rho) = \frac{a^4 - S^2\alpha\rho}{a^4}.$$

$$\therefore\ SY \,.\, HZ = a^2(1 - e^2) = b^2.$$

Ex. 3. *The perpendicular from the focus on the tangent intersects the tangent in the circumference of the circle described about the axis major.*

Retaining the notation of the last example, we have

$$CY = \alpha + x\phi\rho$$

$$= \alpha + \frac{\phi\rho\,(1 - S\alpha\phi\rho)}{(\phi\rho)^2};$$

$$\therefore\ CY^2 = a^2 + \frac{2S\alpha\phi\rho\,(1 - S\alpha\phi\rho)}{(\phi\rho)^2} + \frac{(1 - S\alpha\phi\rho)^2}{(\phi\rho)^2}$$

$$= a^2 + \frac{1 - S^2\alpha\phi\rho}{(\phi\rho)^2}$$

$$= -a^2e^2 - a^2(1 - e^2) \text{ (last example)}$$

$$= -a^2,$$

and the line $CY = a$.

Ex. 4. *To find the locus of T when the perpendicular from the centre on the chord of contact is constant.*

If CT be π, the equation of QR, the chord of contact, is

$$S\sigma\phi\pi = 1 \text{ (Art. 49. 3)},$$

and the perpendicular (Ex. 1) is $T\dfrac{1}{\phi\pi}$;

$$\therefore\ (\phi\pi)^2 = -c^2,$$

$$\text{or } S\phi\pi\,.\,\phi\pi = -c^2,$$

$$\text{or } S\pi\phi\phi\pi = -c^2 \text{ (Art. 44. 3)};$$

$$\text{i.e. } S\pi\left(\frac{iSi\pi}{a^4} + \frac{jSj\pi}{b^4}\right) = c^2 \text{ (47. 3)},$$

$$\text{or } \frac{x^2}{a^4} + \frac{y^2}{b^4} = c^2,$$

an ellipse.

Ex. 5. *TQ, TR are two tangents to an ellipse, and CQ', CR' are drawn to the ellipse parallel respectively to TQ, TR; prove that $Q'R'$ is parallel to QR.*

Let $\qquad CQ = \rho,\ CR = \rho',\ CT = \alpha,$

then

$$S\rho\phi\alpha = 1,$$

$$S\rho'\phi\alpha = 1.$$

Now since CQ' is parallel to TQ,

$$CQ' = xTQ = x(\rho - \alpha).$$

Similarly $$CR' = y(\rho' - \alpha),$$

and $$S\,.\,CQ'\phi(CQ') = 1$$

gives $$x^2 S(\rho - \alpha)\,\phi(\rho - \alpha) = 1,$$

$$\text{i. e. } x^2(S\alpha\phi\alpha - 1) = 1,$$

and $$y^2(S\alpha\phi\alpha - 1) = 1;$$

$$\therefore\ y = x,$$

and $$Q'R' = CR' - CQ' = x(\rho' - \rho)$$
$$= xQR;$$

hence $Q'R'$ is parallel to QR.

COR. $$Q'R'^2 : QR^2 :: x^2 : 1$$
$$:: 1 : S\alpha\phi\alpha - 1$$
$$:: 1 : \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1,$$

where x, y are the co-ordinates of T.

Ex. 6. *If a parallelogram be inscribed in an ellipse, its sides are parallel to conjugate diameters.*

Let $PQRS$ be the parallelogram.

$$PQ = \alpha,\ PS = \beta,$$
$$CP = \rho,\ CS = \rho';$$

then $$CQ = \rho + \alpha,\ CR = \rho' + \alpha;$$

$$\therefore\ S\rho\phi\rho = 1,$$
$$S(\rho + \alpha)\,\phi(\rho + \alpha) = 1;$$

wherefore $$2S\rho\phi\alpha + S\alpha\phi\alpha = 0.$$

Similarly $$2S\rho'\phi\alpha + S\alpha\phi\alpha = 0;$$

$$\therefore\ S(\rho' - \rho)\,\phi\alpha = 0, \text{ by subtraction,}$$
$$\text{or } S\beta\phi\alpha = 0,$$

and (48. Cor.) β, α are parallel to conjugate diameters.

Additional Examples to Chap. VI.

1. Shew that the locus of the points of bisection of chords to an ellipse, all of which pass through a given point, is an ellipse.

2. The locus of the middle points of all straight lines of constant length terminated by two fixed straight lines, is an ellipse whose centre bisects the shortest distance between the fixed lines; and whose axes are equally inclined to them.

3. If chords to an ellipse intersect one another in a given point, the rectangles by their segments are to one another as the squares of semi-diameters parallel to them.

4. If PCP', DCD' are conjugate diameters, then PD, PD' are proportional to the diameters parallel to them.

5. If Q be a point in the focal distance SP of an ellipse, such that SQ is to SP in a constant ratio, the locus of Q is a similar ellipse.

6. Diameters which coincide with the diagonals of the parallelogram on the axes are equal and conjugate.

7. Also diameters which coincide with the diagonals of any parallelogram formed by tangents at the extremities of conjugate diameters are conjugate.

8. The angular points of these parallelograms lie on an ellipse similar to the given ellipse and of twice its area.

9. If from the extremities of the axes of an ellipse four parallel lines be drawn, the points in which they cut the curve are the extremities of conjugate diameters.

10. If from the extremity of each of two semi-diameters ordinates be drawn to the other, the two triangles so formed will be equal in area.

11. Also if tangents be drawn from the extremity of each to meet the other produced, the two triangles so formed will be equal in area.

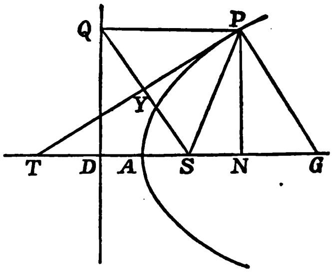
12. If on the semi-axes a parallelogram be described, and about it an ellipse similar and similarly situated to the given ellipse be constructed, any chord PQR of the larger ellipse, drawn from the further extremity of the diameter CD of the smaller ellipse, is bisected by the smaller ellipse at Q.

13. If TP, TQ be tangents to an ellipse, and PCP' be the diameter through P, then $P'Q$ is parallel to CT.

CHAPTER VII.

THE PARABOLA AND HYPERBOLA.

51. As already stated, most of the properties of the hyperbola are the same as the corresponding properties of the ellipse, and proved by the same process, e being greater than 1. There are, however, some properties both of it and of the parabola which may be conveniently developed by a process more analogous to that of the Cartesian geometry. This process we shall develope presently. In the meantime we proceed to give a brief outline of the application to the parabola of the method employed in the preceding Chapter for the ellipse.



52. If S be the focus of a parabola, DQ the directrix, we have $SP = PQ$, $SA = AD = a$.

If $SP = \rho$, $SD = \alpha$, we have (Ex. 5, Art. 35)

$$\alpha^2\rho^2 = (\alpha^2 - S\alpha\rho)^2 \quad \ldots\ldots\ldots (1).$$

If $$\phi\rho = \frac{\rho - \alpha^{-1}S\alpha\rho}{\alpha^2} \quad \ldots\ldots (2),$$

to which the properties of $\phi\rho$ in Art. 44 evidently apply, the equation becomes

$$S\rho\,(\phi\rho + 2\alpha^{-1}) = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

If ρ' be another point in the parabola, $\rho' - \rho = \beta$, the limit to which β approaches is a vector along the tangent; so that if $x\beta = \varpi - \rho$, ϖ is the vector to a point in the tangent; this gives

$$S(\varpi - \rho)\,(\phi\rho + \alpha^{-1}) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4);$$

hence the equation of the tangent becomes

$$S\pi(\phi\rho + \alpha^{-1}) + S\alpha^{-1}\rho = 1 \qquad (5).$$

From (2) it is evident that

$$S\alpha\phi\rho = 0 \qquad (6),$$

so that $\phi\rho$ is a vector perpendicular to the axis.

From the same equation

$$S\rho\phi\rho = \frac{\rho^2 - \alpha^{-2}S^2\alpha\rho}{a^2}$$

$$= \frac{(\rho - \alpha^{-1}S\alpha\rho)^2}{a^2}$$

$$= a^2(\phi\rho)^2 \qquad (7).$$

From (4) the normal vector is

$$\phi\rho + \alpha^{-1} \qquad (8);$$

therefore the equation of the normal is

$$\sigma = \rho + x(\phi\rho + \alpha^{-1}) \qquad (9).$$

Equation (2) when exhibited as

$$a^2\phi\rho = \rho - \alpha^{-1}S\alpha\rho,$$

reads by (6), 'vector along $NP = SP$ - vector along AN', which requires that

$$NP = a^2\phi\rho \qquad (10),$$

$$SN = \alpha^{-1}S\alpha\rho;$$

$$\text{i. e. } = \alpha S\alpha^{-1}\rho \qquad (11).$$

For the subtangent AT, put $x\alpha$ for π in (5), and there results by (6)

$$x + S\alpha^{-1}\rho = 1,$$

whence

$$\left(x - \frac{1}{2}\right)\alpha = \frac{1}{2}\alpha - \alpha S\alpha^{-1}\rho;$$

i. e. vector $AT = -$ vector AN (by 11);

$\therefore$ line $AT = AN$;

and $$ST = x\alpha \text{ gives}$$

$$ST^2 = (\alpha - \alpha S\alpha^{-1}\rho)^2$$

$$= \frac{(\alpha^2 - S\alpha\rho)^2}{\alpha^2}$$

$$= \rho^2 \text{ by (1)};$$

$$\therefore \text{ line } ST = SP,$$

whence also the tangent bisects the angle SPQ; and SQ is perpendicular to and bisected by the tangent.

From (8) $$y(\phi\rho + \alpha^{-1}) = PG$$

$$= PN + NG$$

$$= -\alpha^2\phi\rho + z\alpha \text{ (by 10)};$$

$$\therefore y = -\alpha^2, \quad y = z\alpha^2,$$

$$z = -1,$$

$$z\alpha = -\alpha;$$

$$\text{i. e. } NG = -SD,$$

$$\text{or line } NG = SD,$$

whence the subnormal is constant.

And $$\text{vector } GP = -y(\phi\rho + \alpha^{-1}) = \alpha^2(\phi\rho + \alpha^{-1});$$

$$\therefore \text{ vector } SQ = SD + DQ$$

$$= SD + NP$$

$$= \alpha + \alpha^2\phi\rho$$

$$= GP,$$

and $SQGP$ is a rhombus.

Lastly, $$\frac{1}{2}(\alpha + \alpha^2\phi\rho) = \frac{1}{2}SQ$$

$$= SY$$

$$= SA + AY;$$

$$\therefore AY = \frac{1}{2}\alpha^2\phi\rho;$$

or (10) AY is parallel to, and equal to half of NP.

53. If now we substitute Cartesian coordinates, making

$$\rho = xi + yj, \quad \alpha = -2ai;$$

we shall have

$$S\alpha^{-1}\rho = -\frac{x}{2a},$$

$$\alpha^{-1}S\alpha\rho = xi,$$

$$\phi\rho = -\frac{yj}{4a^2};$$

and equation (3) becomes

$$\frac{y^2}{4a^2} - \frac{x}{a} = 1,$$

$$\text{or} \quad y^2 = 4a(a + x)$$

$$= 4ax' \text{ if } x' = AN.$$

The locus of the middle points of parallel chords is thus found.

Let the chords be parallel to β, π the vector of the middle point of one of the chords,

then $$\pi + x\beta = \rho,$$

and $$S(\pi + x\beta)\,\phi(\pi + x\beta) + 2S\alpha^{-1}(\pi + x\beta) = 1;$$

which, since the term involving x must disappear, gives

$$S\pi\phi\beta + S\alpha^{-1}\beta = 0,$$

a straight line perpendicular to $\phi\beta$, i.e. (6) parallel to the axis.

This equation may be written

$$S\beta(\phi\pi + \alpha^{-1}) = 0,$$

which shews (8) that the chords are perpendicular to the normal vector at the point where $\rho = \pi$, i.e. at the point where the locus of the chords meets the curve: in other words, the chords are parallel to the tangent at the extremity of the diameter which bisects them.

54. Examples.

Ex. 1. *If two chords be drawn always parallel to given lines, and cut one another at points either within or without the parabola,*

the ratio of the rectangles of their segments is always the same whatever be their point of section.

Let POp, QOq be the chords drawn through O, and always parallel respectively to β and γ, which we will suppose to be unit vectors.

Let δ be the vector to O,

then
$$\rho = \delta + x\beta$$

gives from equation (3)

$$S(\delta + x\beta)(\phi\delta + \phi x\beta + 2a^{-1}) = 1;$$

$$\therefore\ x^2 S\beta\phi\beta + S\delta\phi\delta + 2Sa^{-1}\delta + Ax = 1,$$

the product of the two values of x being

$$\frac{S\delta\phi\delta + 2Sa^{-1}\delta - 1}{S\beta\phi\beta};$$

$$\therefore\ OP.Op : OQ.Oq :: \frac{1}{S\beta\phi\beta} : \frac{1}{S\gamma\phi\gamma}$$

a constant ratio whatever be O.

COR. Let θ, θ' be the angles in which β and γ cut the axis; then since β, γ are unit vectors, if ρ be a vector to the parabola, drawn from S parallel to POp, which we may now call SP;

$$\rho = n\beta,\quad \phi\rho = \phi(n\beta) = n\phi\beta \ (44.\ 2),$$

will give

$$S\beta\phi\beta = \frac{S\rho\phi\rho}{n^2} = \frac{S\rho\phi\rho}{SP^2},$$

in which case
$$\phi\rho \text{ is } \frac{NP}{a^2};$$

$$\therefore\ S\beta\phi\beta : S\gamma\phi\gamma :: \sin\theta\frac{NP}{SP} : \sin\theta'\frac{N'P'}{SP'} :: \sin^2\theta : \sin^2\theta';$$

$$\text{and, } OP.Op : OQ.Oq :: \frac{1}{\sin^2\theta} : \frac{1}{\sin^2\theta'}.$$

EX. 2. *Find the locus of the point which divides a system of parallel chords into segments whose product is constant.*

By the last example, the equation of the locus is

$$\frac{S\delta\phi\delta + 2S\alpha^{-1}\delta - 1}{S\beta\phi\beta} = c,$$

a parabola similar to the given parabola.

Ex. 3. *The perpendicular from A on the tangent, and the line PQ are produced to meet in R: find the locus of R.*

By Art. 52. 8, $AR = x(\phi\rho + \alpha^{-1})$,

and $PR = y\alpha$;

$$\therefore \ \frac{\alpha}{2} + x(\phi\rho + \alpha^{-1}) = \rho + y\alpha = \pi.$$

Operate by $S\phi\rho$,

and
$$x(\phi\rho)^2 = S\rho\phi\rho$$
$$= a^2(\phi\rho)^2 \ \ (52.\ 7);$$
$$\therefore \ x = a^2,$$

and
$$\pi = \frac{\alpha}{2} + a^2(\phi\rho + \alpha^{-1})$$
$$= \frac{3\alpha}{2} + a^2\phi\rho \text{ is the equation required;}$$

and, since $S\left(\pi - \frac{3\alpha}{2}\right)\alpha = 0$, it is that of a straight line perpendicular to the axis, at the distance $3a$ from S.

Ex. 4. *To find the locus of the intersection with the tangent of the perpendicular on it from the vertex.*

If π be the vector perpendicular on the tangent from A, we have by (52. 8)

$$\pi = x(\phi\rho + \alpha^{-1}) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and the equation of the tangent gives, putting $\pi + \frac{\alpha}{2}$ in place of π in (52. 5), and multiplying by 2,

$$2S\pi\phi\rho + 2S\alpha^{-1}\pi + 2S\alpha^{-1}\rho = 1 \ \ldots\ldots\ldots\ldots(2),$$

we have also

$$S\rho(\phi\rho + 2\alpha^{-1}) = 1 \ \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

From these three equations we have to eliminate x and ρ.

Equation (1) gives

$$S\alpha\pi = x,$$

which gives x,

and $$S\pi\phi\rho = x\,(\phi\rho)^2,$$

which substituted in (2) gives

$$2x\,(\phi\rho)^2 + 2S\alpha^{-1}\pi + 2S\alpha^{-1}\rho = 1.$$

Also, substituting (52. 7) $a^2(\phi\rho)^2$ for $S\rho\phi\rho$, equation (3) gives

$$a^2(\phi\rho)^2 + 2S\alpha^{-1}\rho = 1;$$

therefore by subtraction

$$(2x - a^2)(\phi\rho)^2 + 2S\alpha^{-1}\pi = 0,$$

$$\text{i.e. } (2S\alpha\pi - a^2)(\phi\rho)^2 + 2S\alpha^{-1}\pi = 0,$$

which from (1) becomes, multiplying by $S^2\alpha\pi$,

$$(2S\alpha\pi - \alpha)^2(\pi - \alpha^{-1}S\alpha\pi)^2 + 2S^2\alpha\pi S\alpha^{-1}\pi = 0.$$

This equation at once reduces to

$$2\pi^2 S\alpha\pi - \pi^2\alpha^2 + S^2\alpha\pi = 0,$$

an equation which, when 4α is written in place of α, becomes identical with that obtained in Art. 37. Ex. 8.

The locus is therefore a cissoid, the diameter of the generating circle being AD.

55. It will probably have suggested itself to the reader, that there exists a large class of problems to which the processes we have illustrated are scarcely if at all applicable. Hence there may have arisen a contrast between the Cartesian Geometry and Quaternions unfavourable to the latter. To remove this unfavourable impression, all that is required in a reader familiar with the older Geometry is a little experience in combining the logic of the new analysis with the forms of the old. He will then see how simple and direct are the arguments which he can bring to bear on any individual problem, and consequently how little the memory is taxed.

We propose in this Article to put the reader in the track of employing his old forms in conjunction with quaternion reasonings.

We shall work several examples on the parabola and the hyperbola. Having applied quaternions pretty fully to the ellipse in what has preceded, we will limit ourselves to a single example in this case.

1. *The Parabola.* If the unit vector along any diameter of the parabola be α, and the unit vector parallel to the tangent at its extremity be β; we may write the equation of the parabola under the form

$$\begin{aligned}\rho &= x\alpha + y\beta \\ &= \frac{y^2}{4a'}\alpha + y\beta \quad \text{.............................(1).}\end{aligned}$$

For the particular case in which the diameter in question is the axis, and the tangent at its extremity parallel to the directrix

$$\rho = \frac{y^2}{4a}\alpha + y\beta \quad \text{..........................(2),}$$

where a is AS (Art. 52).

This is the most convenient form when the focus is referred to.

In other cases a somewhat simpler form may be obtained by supposing α, or if necessary both α and β of equation (1) to be other than unit vectors.

The equation may then be written under the form

$$\rho = \frac{t^2}{2}\alpha + t\beta \quad \text{..........................(3).}$$

To find the equation of the tangent, we have

$$\rho' = \frac{t'^2}{2}\alpha + t'\beta;$$

$$\therefore\ \rho' - \rho = (t' - t)\left(\frac{t' + t}{2}\alpha + \beta\right).$$

Now $\rho' - \rho$ is a vector along the secant; and its limit is a vector along the tangent: hence any vector along the tangent is a multiple of $t\alpha + \beta$; and the equation of the tangent may be written

$$\varpi = \frac{t^2}{2}\alpha + t\beta + x(t\alpha + \beta) \text{ .... } (4).$$

EXAMPLES.

Ex. 1. *If AP, AQ be chords drawn at right angles to one another from A; PM, QN perpendiculars on the axis, then the latus rectum is a mean proportional between AM and AN; or between PM and QN.*

If $PM = y$, $QN = y'$,

$$AP = \frac{y^2}{4a}\alpha + y\beta, \quad AQ = \frac{y'^2}{4a}\alpha - y'\beta.$$

Now $S(AP \,.\, AQ) = 0$ (22. 7);

$$\therefore \frac{y^2 y'^2}{(4a)^2} - yy' = 0,$$

$$\text{or } yy' = (4a)^2;$$

therefore also $xx' = (4a)^2$.

Ex. 2. *If the rectangle of which AP, AQ are the sides be completed, the further angle will trace out a parabola similar to the given parabola, the distance between the two vertices being equal to twice the latus rectum.*

$$\begin{aligned} \rho &= AP + AQ \\ &= \frac{y^2 + y'^2}{4a}\alpha + (y - y')\beta \\ &= \frac{(y - y')^2}{4a}\alpha + (y - y')\beta + 8a\alpha. \end{aligned}$$

Ex. 3. *The circle described on a focal chord as diameter touches the directrix; and the circle described on any other chord does not reach the directrix.*

Let PQ be *any* chord, centre O,

$$AP = \frac{y^2}{4a}\alpha + y\beta, \quad AQ = \frac{y'^2}{4a}\alpha + y'\beta.$$

The equation of the circle with centre O, radius OP, is

$$\left(\rho - \frac{AQ + AP}{2}\right)^2 = \left(\frac{AQ - AP}{2}\right)^2,$$

$$\text{or } \rho^2 - S(AP + AQ)\rho + S(AP \,.\, AQ) = 0.$$

At the points in which this circle meets the directrix

$$\rho = -a\alpha + z\beta;$$

$$\therefore \; -a^2 - z^2 - \frac{y^2 + y'^2}{4} + z(y + y') - \frac{y^2 y'^2}{(4a)^2} - yy' = 0,$$

or
$$z^2 - z(y + y') + \frac{(y + y')^2}{4} = -\left(\frac{yy'}{4a} + a\right)^2.$$

This equation is possible only when

$$yy' + 4a^2 = 0;$$

i.e. when the chord is a focal chord.

In this case the two values of z are equal, each being $\frac{y + y'}{2}$; and the directrix is a tangent to the circle.

Ex. 4. *Two parabolas have a common focus and axis; their vertices are turned in opposite directions. A focal chord cuts them in PQ, $P'Q'$, so that $PP'SQQ'$ are in order. Prove* (1) *that $SP \,.\, SP' = SQ \,.\, SQ'$;* (2) *that $SP : SQ'$ is a constant ratio; and* (3) *that the tangents at P, P' are at right angles to one another.*

The equations of the parabolas are

$$\rho = -a\alpha + \frac{y^2}{4a}\alpha + y\beta,$$

$$\rho' = a'\alpha - \frac{y'^2}{4a'}\alpha + y'\beta,$$

the focus being the origin.

Now since ρ, ρ' are in the same straight line when the common chord is the focal chord, we have

$$\rho' = p\rho;$$

$$\therefore\ a' - \frac{y'^2}{4a'} = -pa + \frac{py^2}{4a},$$

$$y' = py,$$

$$\therefore\ (yy' - 4aa')(a'y + ay') = 0.$$

Taking the former factor, we must have y, y' on the same side of the axis with a constant product; therefore

$$SP \,.\, SP' = SQ \,.\, SQ'.$$

The second factor gives $SP : SQ'$ a constant ratio $a : a'$.

Lastly, by Equation (4), the tangent vectors at P and P' are parallel to

$$\frac{y}{2a}\alpha + \beta, \quad -\frac{y'}{2a'}\alpha + \beta.$$

Now $$S\left(\frac{y}{2a}\alpha + \beta\right)\left(-\frac{y'}{2a'}\alpha + \beta\right) = \frac{yy'}{4aa'} - 1 = 0;$$

therefore the tangents are at right angles to one another.

Ex. 5. *If a triangle be inscribed in a parabola, the three points in which the sides are met by the tangents at the angles lie in a straight line.*

Let OPQ be the triangle.

Take O as the origin, then

$$\rho = \frac{t^2}{2}\alpha + t\beta,$$

$$\rho' = \frac{t'^2}{2}\alpha + t'\beta,$$

$$\varpi = \frac{t^2}{2}\alpha + t\beta + x(t\alpha + \beta),$$

$$\varpi' = \frac{t'^2}{2}\alpha + t'\beta + x'(t'\alpha + \beta),$$

are the vectors OP, OQ, and the equations of the tangents at P and Q.

If QO meet in A the tangent at P,

$$\begin{aligned} OA &= \frac{t^2}{2}\alpha + t\beta + x(t\alpha + \beta) \\ &= yOQ \\ &= y\left(\frac{t'^2}{2}\alpha + t'\beta\right); \end{aligned}$$

$$\therefore \frac{t^2}{2} + tx = \frac{t'^2}{2}y,$$

$$t + x = t'y,$$

$$y = \frac{t^2}{2tt' - t'^2},$$

and
$$\begin{aligned} OA &= \frac{t^2}{2tt' - t'^2}\left(\frac{t'^2}{2}\alpha + t'\beta\right) \\ &= \frac{t^2}{2t - t'}\left(\frac{t'}{2}\alpha + \beta\right). \end{aligned}$$

Similarly if the tangent at Q meets PO in B,

$$OB = \frac{t'^2}{2t' - t}\left(\frac{t}{2}\alpha + \beta\right).$$

If the tangent at O meets PQ in C,

$$\begin{aligned} OC &= OP + z(PQ) \\ &= OP + z(OQ - OP) \\ &= \frac{t^2}{2}\alpha + t\beta + z\left\{\frac{t'^2 - t^2}{2}\alpha + (t' - t)\beta\right\}. \end{aligned}$$

But
$$OC = v\beta;$$

$$\therefore \frac{t^2}{2} + z\frac{t'^2 - t^2}{2} = 0,$$

$$t + z(t' - t) = v,$$

$$v = \frac{tt'}{t + t'},$$

and $$OC = \frac{tt'}{t+t'}\beta.$$

Now $$\frac{2t-t'}{t}OA - \frac{2t'-t}{t'}OB - \frac{t^2-t'^2}{tt'}OC = 0,$$

and also $$\frac{2t-t'}{t} - \frac{2t'-t}{t'} - \frac{t^2-t'^2}{tt'} = 0;$$

therefore (Art. 13) A, B, C are in a straight line.

2. *The ellipse.* If α, β are unit vectors along the axes, the equation of the ellipse may be written

$$\rho = x\alpha + y\beta,$$

where $$y^2 = \frac{b^2}{a^2}(a^2 - x^2) = m(a^2 - x^2);$$

and the equation of the tangent will be readily seen to be

$$\varpi = x\alpha + y\beta + X(y\alpha - mx\beta).$$

A single example will suffice.

Ex. *If tangents be drawn at three points* P, Q, R *of an ellipse intersecting in* R′, Q′, P′, *prove that*

$$PR'.QP'.RQ' = PQ'.QR'.RP'.$$

If x, y; x', y'; x'', y'' are respectively the co-ordinates of P, Q, R; we shall have

$$CR' = x\alpha + y\beta + X(y\alpha - mx\beta)$$
$$= x'\alpha + y'\beta + X'(y'\alpha - mx'\beta);$$
$$\therefore\ x + Xy = x' + X'y',$$
$$y - mXx = y' - mX'x';$$
$$\therefore\ mX(x'y - y'x) = mx'^2 + y'^2 - mxx' - yy'$$
$$= b^2 - mxx' - yy'.$$

Hence $$mX'(xy' - x'y) = b^2 - mxx' - yy'$$
$$= -mX(xy' - x'y);$$
$$\therefore\ X = -X',$$
$$Y = -Y' \text{ for } Q',$$
$$Z = -Z' \text{ for } P',$$

and $$XY'Z = -X'YZ'.$$

Now $$\frac{X}{Y} = \frac{PR'}{PQ'}, \text{ \&c.}$$
hence the proposition.

3. *The hyperbola.* If α, β are unit vectors parallel to the asymptotes CX, CY, the equation of the hyperbola may be written

$$\rho = x\alpha + y\beta$$
$$= x\alpha + \frac{C}{x}\beta,$$

since $$xy = \frac{a^2 + b^2}{4} = C.$$

If α, β be not both units we may write the equation under the simpler form

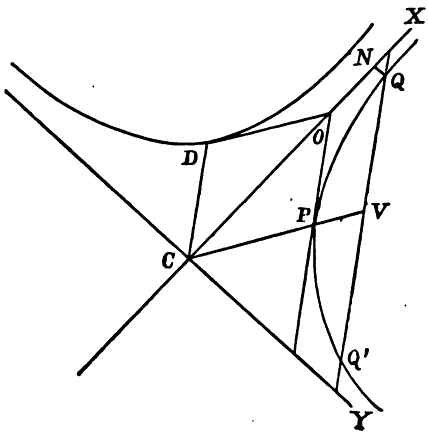
$$\rho = t\alpha + \frac{\beta}{t} \qquad \text{(1).}$$

To find the equation of the tangent, we have as usual a vector parallel to the secant

$$= \rho' - \rho = (t' - t)\left(\alpha - \frac{\beta}{tt'}\right)$$
$$= \left(\frac{t' - t}{t}\right)\left(t\alpha - \frac{\beta}{t'}\right),$$

and a vector parallel to the tangent will be

$$t\alpha - \frac{\beta}{t} \qquad \text{(2).}$$



Hence the equation of the tangent is

$$\pi = ta + \frac{\beta}{t} + x\left(ta - \frac{\beta}{t}\right) \text{ ....(3).}$$

COR. It is evident that

$$ta + \frac{\beta}{t}, \quad ta - \frac{\beta}{t},$$

are conjugate semi-diameters.

EXAMPLES.

Ex. 1. *One diagonal of a parallelogram whose sides are the co-ordinates being the radius vector, the other diagonal is parallel to the tangent.*

We have $$CN = ta, \; NQ = \frac{\beta}{t},$$

$$CQ = ta + \frac{\beta}{t},$$

and the other diagonal is

$$ta - \frac{\beta}{t},$$

which, equation (2), is parallel to the tangent at Q.

Ex. 2. *Any diameter CP bisects all the chords which are parallel to the tangent at P.*

Let CP be $$ta + \frac{\beta}{t},$$

then the tangent at P is parallel to

$$ta - \frac{\beta}{t};$$

$$\therefore \; CQ = CV + VQ = X\left(ta + \frac{\beta}{t}\right) + Y\left(ta - \frac{\beta}{t}\right).$$

But as Q is a point in the hyperbola, this equation must have the form

$$CQ = Ta + \frac{\beta}{T};$$

$$\therefore\ (X+Y)\,t=T,$$

$$(X-Y)\,\frac{1}{t}=\frac{1}{T},$$

and
$$X^2-Y^2=1,$$

an equation which gives two equal values of Y with opposite signs, for every value of X.

Hence all chords are bisected.

COR. $X^2-Y^2=1$ is

$$\left(\frac{CV}{CP}\right)^2-\left(\frac{QV}{CD}\right)^2=1,$$

CD being
$$t\alpha-\frac{\beta}{t}=PO.$$

This is the ordinary equation of the hyperbola referred to conjugate diameters.

Ex. 3. *If TQ, $T'Q'$ be two tangents to the hyperbola intersecting in R and terminated at T, T', Q, Q' by the asymptotes; then* (1) *TQ' is parallel to $T'Q$;* (2) *area of triangle TRT' = area of triangle QRQ', and* (3) *CR bisects TQ' and $T'Q$.*

The equation of the tangent

$$\pi=t\alpha+\frac{\beta}{t}+x\left(t\alpha-\frac{\beta}{t}\right),$$

gives
$$CT=2t\alpha,$$

(the coefficient of β being 0),

$$CQ=\frac{2\beta}{t},$$

$$CT'=2t'\alpha,$$

$$CQ'=\frac{2\beta}{t'};$$

$$\therefore\ Q'T=2\alpha t-\frac{2\beta}{t'}=\frac{2}{t'}(\alpha tt'-\beta),$$

$$QT'=\frac{2}{t}(\alpha tt'-\beta);$$

therefore $Q'T$ is parallel to QT'.

Again, $$CR = CQ + QR = CQ + x\,(CT - CQ)$$
$$= \frac{2\beta}{t} + x2\left(at - \frac{\beta}{t}\right).$$

Also $$CR = \frac{2\beta}{t'} + x'2\left(at' - \frac{\beta}{t'}\right);$$
$$\therefore\ xt = x't',$$
$$\frac{1}{t} - \frac{x}{t} = \frac{1}{t'} - \frac{x'}{t'},$$
$$x = \frac{t'}{t + t'},$$
$$x' = \frac{t}{t + t'},$$

and $$xx' = (1 - x)(1 - x'),$$
$$\text{i.e. } QR\,.\,Q'R = RT\,.\,R'T,$$

and the triangles TRT', QRQ' are equal.

Lastly, $$CR = \frac{2\beta}{t} + \frac{2t'}{t + t'}\left(at - \frac{\beta}{t}\right)$$
$$= \frac{t'}{t + t'}\left(2ta + \frac{2\beta}{t'}\right),$$

or CR is in the direction of the diagonal of the parallelogram of which the sides are CT, CQ'; and therefore CR bisects TQ' and $T'Q$.

Ex. 4. *If through Q, P, Q' parallels be drawn to CX meeting CY in E, F, G; CE, CF, CG are in continued proportion.*

$$CP = ta + \frac{\beta}{t};$$
$$Q'Q = m\left(ta - \frac{\beta}{t}\right);$$
$$\therefore\ CQ = CV + VQ$$
$$= X\left(ta + \frac{\beta}{t}\right) + Y\left(ta - \frac{\beta}{t}\right),$$

$$CQ' = X\left(t\alpha + \frac{\beta}{t}\right) - Y\left(t\alpha - \frac{\beta}{t}\right),$$

$$CE = (X - Y)\frac{\beta}{t},$$

$$CF = \frac{\beta}{t},$$

$$CG = (X + Y)\frac{\beta}{t};$$

and $$CE \,.\, CG = CF^2;$$

because $$X^2 - Y^2 = 1 \text{ (Ex. 2).}$$

Ex. 5. *If a chord of a hyperbola be one diagonal of a parallelogram whose sides are parallel to the asymptotes, the other diagonal passes through the centre.*

Let the chord be PQ; ρ, ρ' the vectors to P and Q; then

$$QP = \rho - \rho' = \alpha t + \frac{\beta}{t} - \left(\alpha t' + \frac{\beta}{t'}\right).$$

Now when one diagonal of a parallelogram is $m\alpha + n\beta$, the other will be $m\alpha - n\beta$.

Therefore in the case before us, the other diagonal is

$$\alpha(t - t') - \beta\left(\frac{1}{t} - \frac{1}{t'}\right)$$

$$= (t - t')\left(\alpha + \frac{\beta}{tt'}\right)$$

$$= \frac{t - t'}{t + t'}\left\{\alpha(t + t') + \beta\left(\frac{1}{t} + \frac{1}{t'}\right)\right\}$$

$$= \frac{t - t'}{t + t'}(\rho + \rho').$$

And it is therefore in the same straight line with the line which joins the centre of the hyperbola with the middle point of PQ; whence the truth of the proposition.

Ex. 6. *If two tangents to a hyperbola at the extremities Q, Q' of a diameter, meet a tangent at P in the points T, T'; and if CD, CD' are the semi-diameters conjugate to CP, CQ; then* (1) $PT : QT :: PT' : Q'T' :: CD : CD'$;
and (2) $PT \,.\, PT' = CD^2$.

If t, t', $-t'$, correspond to P, Q, Q', then

$$CT = at + \frac{\beta}{t} + x\left(at - \frac{\beta}{t}\right)$$

$$= at' + \frac{\beta}{t'} + x'\left(at' - \frac{\beta}{t'}\right),$$

gives
$$t + xt = t' + x't',$$

$$\frac{1}{t} - \frac{x}{t} = \frac{1}{t'} - \frac{x'}{t'},$$

$$x = \frac{t'-t}{t'+t} = -x'.$$

Similarly
$$CT' = at + \frac{\beta}{t} + y\left(at - \frac{\beta}{t}\right)$$

$$= -at' - \frac{\beta}{t'} - y'\left(at' - \frac{\beta}{t'}\right),$$

gives
$$t + yt = -t' - y't',$$

$$\frac{1}{t} - \frac{y}{t} = -\frac{1}{t'} + \frac{y'}{t'},$$

whence
$$y = \frac{t'+t}{t'-t} = -y'.$$

Now
$$x : y :: x' : y'$$

gives
$$PT : QT :: PT' : QT'$$
$$:: CD : CD'.$$

And
$$xy = 1$$

gives
$$PT \,.\, PT' = C'D^2.$$

COR.
$$x'y' = 1,$$

gives
$$QT \,.\, Q'T' = CD'^2.$$

Ex. 7. *Straight lines move so that the triangular area which they cut off from two given straight lines which meet one another is constant: to find the locus of their ultimate intersections.*

Let OAA', OBB' be the fixed lines, AB, $A'B'$ two of the moving lines with the condition that

$$OA \,.\, OB = OA' \,.\, OB'.$$

If α, β be unit vectors along OA, OB,

$$OA = t\alpha,\;\; OB = u\beta\,;\;\; OA' = t'\alpha,\;\; OB' = u'\beta,$$

the point of intersection of AB, AB' gives

$$\rho = t\alpha + x\,(u\beta - t\alpha)$$
$$= t'\alpha + x'\,(u'\beta - t'\alpha),$$
$$\therefore\;\; xu = x'u',$$

and
$$t\,(1 - x) = t'\,(1 - x')$$
$$= t'\left(1 - \frac{xu}{u'}\right).$$

Now $tu = t'u' = c$ because the triangle has a constant area;

$$\therefore\;\; x = \frac{t}{t + t'} = \frac{1}{2} \text{ ultimately};$$

$$\therefore\;\; \rho = \frac{1}{2}\,t\alpha + \frac{1}{2}\,u\beta = \frac{1}{2}\,t\alpha + \frac{1}{2}\,\frac{c\beta}{t},$$

the equation of a hyperbola.

Additional Examples to Chap. VII.

1. In the parabola $SY^2 = SP \,.\, SA$.

2. If the tangent to a parabola cut the directrix in R, SR is perpendicular to SP.

3. A circle has its centre at the vertex A of a parabola whose focus is S, and the diameter of the circle is $3AS$. Prove that the common chord bisects AS.

4. The tangent at any point of a parabola meets the directrix and latus rectum in two points equally distant from the focus.

5. The circle described on SP as diameter is touched by the tangent at the vertex.

6. Parabolas have their axes parallel and all pass through two given points. Prove that their foci lie in a conic section.

7. Two parabolas have a common directrix. Prove that their common chord bisects at right angles the line joining their foci.

8. The portion of any tangent to the parabola between tangents which meet in the directrix subtends a right angle at the focus.

9. If from the point of contact of a tangent to a parabola a chord be drawn, and another line be drawn parallel to the axis meeting the chord, tangent and curve; this line will be divided by them in the same ratio as it divides the chord.

10. The middle points of focal chords describe a parabola whose latus rectum is half that of the given parabola.

11. PSQ is a focal chord of a parabola: PA, QA meet the directrix in y, z. Prove that Pz, Qy are parallel to the axis.

12. The tangent at D to the conjugate hyperbola is parallel to CP.

13. The portion of the tangent to a hyperbola which is intercepted by the asymptotes is bisected at the point of contact.

14. The locus of a point which divides in a given ratio lines which cut off equal areas from the space enclosed by two given straight lines is a hyperbola of which these lines are the asymptotes.

15. The tangent to a hyperbola at P meets an asymptote in T, and TQ is drawn to the curve parallel to the other asymptote. PQ produced both ways meets the asymptotes in R, R': RR' is trisected in P, Q.

16. From any point R of an asymptote, RN, RM are drawn parallel to conjugate diameters intersecting the hyperbola and its conjugate in P and D. Prove that CP and CD are conjugate.

17. The intercepts on any straight line between the hyperbola and its asymptotes are equal.

18. If QQ' meet the asymptotes in R, r,

$$RQ \,.\, Qr = PO^2.$$

19. If the tangent at any point meet the asymptotes in X and Y, the area of the triangle XCY is constant.

CHAPTER VIII.

CENTRAL SURFACES OF THE SECOND ORDER, PARTICULARLY THE ELLIPSOID AND CONE.

56. *The Ellipsoid.* In discussing central surfaces of the second order, we shall speak as if our results were limited to the ellipsoid. That such limitation is not, in most cases, necessarily imposed on us, will be apparent to any one who has a slender acquaintance with ordinary Analytical Geometry. We adopt it in order that our language may have more precision, and that, in some instances, our analysis may have greater simplicity. If the centre be made the origin it is clear that the scalar equation can contain no such term as $ASa\rho$, for the *definition* of a central surface requires that the equation shall be satisfied both by $+\rho$ and by $-\rho$.

If we turn to the equation of the ellipse (Art. 43), we shall see at once that the equation of the ellipsoid must have the form

$$a\rho^2 + bS^2a\rho + 2cSa\rho S\beta\rho + \ldots = 1.$$

Now if, as in the Article referred to, we put

$$\phi\rho = a\rho + baSa\rho + c\,(aS\beta\rho + \beta Sa\rho) + \ldots$$

we shall have

$$\begin{aligned} S\rho\phi\rho &= a\rho^2 + bS^2a\rho + 2cSa\rho S\beta\rho + \ldots \\ &= 1, \end{aligned}$$

the equation required.

It will be seen that, as in Arts. 32, 33, one form of the equation of the straight line was found to coincide exactly with the equation of a plane, so a form of the equation of the ellipse coincides exactly with the equation of the ellipsoid.

It is evident that the three properties of $\phi\rho$ given in Art. 44 are true of $\phi\rho$ in its present form.

57. To find the equation of the tangent plane.

Let a secant plane pass through the point whose vector is ρ; and let ρ' be the vector to any point of section.

Put $\rho' = \rho + \beta$, where β is a vector along the secant plane;

then $$S\rho'\phi\rho' = S(\rho + \beta)\,\phi(\rho + \beta).$$

Hence, observing that (44)

$$\phi(\rho + \beta) = \phi\rho + \phi\beta,$$

and $$S\rho\phi\beta = S\beta\phi\rho,$$

we have $$S\rho'\phi\rho' = S\rho\phi\rho + 2S\beta\phi\rho + S\beta\phi\beta;$$

$$\text{i. e. } 2S\beta\phi\rho + S\beta\phi\beta = 0.$$

Now (45), as the secant plane approaches the tangent plane, the sum of these two expressions approaches in value to the first alone: that is, for the tangent plane, $S\beta\phi\rho = 0$, where β is a vector along that plane.

If π be the vector to a point in the tangent plane,

$$\pi = \rho + x\beta;$$

$$\therefore\ S(\pi - \rho)\,\phi\rho = xS\beta\phi\rho$$

$$= 0,$$

and $$S\pi\phi\rho = S\rho\phi\rho$$

$$= 1$$

is the equation of the tangent plane.

COR. $\phi\rho$ is a vector perpendicular to the tangent plane at the extremity of the vector ρ.

58. If OY be perpendicular from the centre O on the tangent plane; then, since $\phi\rho$ is a vector perpendicular to that plane, $OY = x\phi\rho$ and $Sx(\phi\rho)^2 = 1$, giving

$$OY = T(x\phi\rho) = T\frac{1}{\phi\rho}.$$

Sir W. Hamilton terms $\phi\rho$ the *vector of proximity.*

59. If tangent planes all pass through a fixed point, the curve of contact is a plane curve.

Let T be the fixed point; vector α; ρ the vector to a point of contact.

Then (Art. 57) $$S\alpha\phi\rho = 1;$$

$$\text{i. e. } S\rho\phi\alpha = 1 \quad (44.\ 3),$$

which is the equation in ρ of a plane perpendicular to $\phi\alpha$.

Now $\phi\alpha$ is the normal vector of the point where OT cuts the ellipsoid;

$\therefore$ the curve of contact lies in a plane parallel to the tangent plane at the extremity of the diameter drawn to the given point.

The plane of contact is called the polar plane to the point.

60. Tangent planes are all parallel to a given straight line, to find the curve of contact.

Let α be a vector parallel to the given line; then

$$\pi = \rho + x\alpha$$

is a point in the tangent plane;

$$\therefore\ S(\rho + x\alpha)\phi\rho = 1;$$

and $$S\alpha\phi\rho = 0,$$

or $$S\rho\phi\alpha = 0,$$

the equation of a plane through the origin perpendicular to $\phi\alpha$: that is, the curve of contact lies in a plane through the centre parallel to the tangent plane at the extremity of the diameter which is parallel to the given line.

61. To find the locus of the middle points of parallel chords.

Let each of the chords be parallel to α, π the vector to the middle point of one of them; then $\pi + x\alpha$, $\pi - x\alpha$ are points in the ellipsoid.

From the first,

$$S(\pi + x\alpha)\phi(\pi + x\alpha) = 1 \quad (\text{Art. } 56);$$

$$\text{i.e. } S\pi\phi\pi + 2xS\pi\phi\alpha + x^2S\alpha\phi\alpha = 1.$$

From the second,

$$S\pi\phi\pi - 2xS\pi\phi\alpha + x^2 S\alpha\phi\alpha = 1\,;$$

$$\therefore\ \text{subtracting,}\ S\pi\phi\alpha = 0 \quad (1),$$

i. e. the locus is a plane through the centre perpendicular to $\phi\alpha$, or parallel to the tangent plane at the extremity A of the diameter which is drawn parallel to α.

If we call this the plane BOC, B and C being any points in which it cuts the ellipsoid; and if $OB = \beta$, $OC = \gamma$, we shall have

$$S\beta\phi\alpha = 0,\ S\gamma\phi\alpha = 0,$$

and therefore $\quad S\alpha\phi\beta = 0,$

or α satisfies the equation $\quad S\pi\phi\beta = 0$

of the plane which bisects all chords parallel to OB (Equation 1).

Let AOC be this plane which bisects all chords parallel to OB.

Then, since OC or γ is a vector in it,

$$S\gamma\phi\beta = 0,\ \text{ i. e. }\ S\beta\phi\gamma = 0.$$

But we have already proved that

$$S\gamma\phi\alpha = 0,\ \text{ i. e. }\ S\alpha\phi\gamma = 0,$$

because γ is in the plane BOC;

$\therefore$ by equation (1) α, β both satisfy the equation of the plane $S\pi\phi\gamma = 0$, which is the plane bisecting all chords parallel to γ; that plane is therefore the plane AOB: we are thus presented with three lines OA, OB, OC such that all chords parallel to any one of them are bisected by the diametral plane which passes through the other two.

We may term these lines *conjugate* semi-diameters, and the corresponding diametral planes *conjugate diametral planes.*

It is evident that the number of conjugate diameters is unlimited.

Cor. We have the following equations:

$$S\alpha\phi\beta = 0 = S\beta\phi\alpha,$$

$$S\beta\phi\gamma = 0 = S\gamma\phi\beta,$$

$$S\alpha\phi\gamma = 0 = S\gamma\phi\alpha \quad (2).$$

They shew that γ is perpendicular to both $\phi\alpha$ and $\phi\beta$, and is therefore a vector perpendicular to their plane; hence, as in 34. 4,

$$\gamma = xV\phi\alpha\phi\beta.$$

In the same way, since $\phi\gamma$ is perpendicular to both α and β, we have

$$\phi\gamma = yV\alpha\beta;$$

or, neglecting tensors, we have the following vector equalities:

$$\gamma = V\phi\alpha\phi\beta, \quad \beta = V\phi\alpha\phi\gamma, \quad \alpha = V\phi\beta\phi\gamma,$$
$$\phi\gamma = V\alpha\beta, \quad \phi\beta = V\alpha\gamma, \quad \phi\alpha = V\beta\gamma \quad (3).$$

Note also

$$y\phi^{-1}V\alpha\beta = xV\phi\alpha\phi\beta,$$

upon which Hamilton founded his solution of linear equations.

62. If as in Art. 47 we write $-\psi\psi\rho$ for $\phi\rho$, $\psi\rho$ being still a vector, the equation of the ellipsoid assumes the form

$$S\rho\psi(\psi\rho) = -1,$$

i. e. (44) $S\psi\rho\psi\rho = -1$

$$(\psi\rho)^2 = -T(\psi\rho)^2 = -1 \text{ ....} (1),$$

which, if we put $\sigma = \psi\rho$, becomes $T\sigma = 1$, the equation of a sphere.

Hence the ellipsoid can be changed into the sphere and *vice versâ*, by a linear deformation of each vector, the operator being the function ψ or its inverse.

The equations

$$S\alpha\phi\beta = 0, \text{ \&c.}$$

now become $\quad S\alpha\psi^2\beta = 0,$

i. e. $S\psi\alpha\psi\beta = 0$, &c., &c. .... (2).

(1) and (2) shew that $\psi\alpha$, $\psi\beta$, $\psi\gamma$ are unit vectors at right angle to one another.

If we term the sphere $T\sigma = 1$ the unit-sphere, we may enunciate this result by saying that the vectors of the unit-sphere which correspond to semi-conjugate diameters form a rectangular system.

63. Let us now take i, j, k unit vectors along the principal axes of x, y, z; then we shall have

$$\rho = xi + yj + zk \quad\text{...........................(1)},$$

$$\therefore Si\rho = -x, \text{ \&c.}$$

so that for the sake of transformations in which it is desirable that the form of ρ should be retained, we may write

$$\rho = -(iSi\rho + jSj\rho + kSk\rho) \quad\text{.....................(2)};$$

and as $\phi\rho$ is a linear and vector function of ρ, its vector portions along the principal axes will be multiples of

$$iSi\rho,\ jSj\rho,\ kSk\rho;$$

we may therefore write

$$\phi\rho = \frac{iSi\rho}{a^2} + \frac{jSj\rho}{b^2} + \frac{kSk\rho}{c^2} \quad\text{..................(3)},$$

the form a^2 having been assumed in order to make the equation

$$S\rho\phi\rho = 1$$

coincide with the Cartesian equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

As $$\phi\rho = -\psi\psi\rho \quad\text{..........................(4)},$$

we require to take $\psi\rho$ so that performing the operation ψ twice on ρ shall give the same result (with a $-$ sign) as performing the operation ϕ once.

Now a comparison of equations (2) and (3) will shew that the latter operation introduces $\frac{1}{a^2}$ &c. into ρ; it is evident therefore that the former operation (ψ) is to introduce $\frac{1}{a}$ &c. or

$$\psi\rho = -\left(\frac{iSi\rho}{a} + \frac{jSj\rho}{b} + \frac{kSk\rho}{c}\right) \quad\text{..................(5)}.$$

It may perhaps be worth while to verify this result. We have

$$\psi\psi\rho = -\left(\frac{iSi\psi\rho}{a} + \frac{jSj\psi\rho}{b} + \frac{kSk\psi\rho}{c}\right)$$

$$= iS\frac{i}{a}\left(\frac{iSi\rho}{a} + \frac{jSj\rho}{b} + \frac{kSk\rho}{c}\right) + \ldots$$

$$= i\frac{i^2Si\rho}{a^2} + \ldots$$

$$= -\left(\frac{iSi\rho}{a^2} + \frac{jSj\rho}{b^2} + \frac{kSk\rho}{c^2}\right)$$

$$= -\phi\rho.$$

$$\phi^2\rho = \phi\phi\rho = \frac{iSi\phi\rho}{a^2} + \frac{jSj\phi\rho}{b^2} + \frac{kSk\phi\rho}{c^2}$$

$$= -\left(\frac{iSi\rho}{a^4} + \frac{jSj\rho}{b^4} + \frac{kSk\rho}{c^4}\right) \qquad (6),$$

$$\phi^{-1}\rho = a^2iSi\rho + b^2jSj\rho + c^2kSk\rho \qquad (7),$$

because $\phi\phi^{-1}\rho$ produces ρ.

$$\psi^{-1}\rho = -(aiSi\rho + bjSj\rho + ckSk\rho) \qquad (8),$$

$$\rho = \psi^{-1}\psi\rho = -(aiSi\psi\rho + bjSj\psi\rho + ckSk\psi\rho) \qquad (9).$$

It is evident that the properties of Art. 44 apply to all these functions.

64. EXAMPLES.

Ex. 1. *Find the point on an ellipsoid, the tangent plane at which cuts off equal portions from the axes.*

Let x, y, z be the co-ordinates of the point, p the portion cut off, then

$$\rho = xi + yj + zk.$$

Now pi, pj, pk are points on the tangent plane;

$$\therefore\ Spi\phi\rho = 1,$$

which gives

$$pSi\left(\frac{iSi\rho}{a^2} + \ldots\right) = 1,$$

or $$\frac{px}{a^2}=1.$$

Similarly $$\frac{py}{b^2}=1,$$

$$\frac{pz}{c^2}=1,$$

$$\frac{x}{a^2}=\frac{y}{b^2}=\frac{z}{c^2}=\frac{1}{p}=\frac{1}{\sqrt{a^2+b^2+c^2}}.$$

Ex. 2. *To find the perpendicular from the centre of the ellipsoid on a tangent plane.*

$$OY^2=\left(T\frac{1}{\phi\rho}\right)^2; \quad \text{(Art. 58)}$$

$$\therefore \frac{1}{OY^2}=(T\phi\rho)^2=-(\phi\rho)^2=\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4} \quad \text{(Art. 63, l. 3)}.$$

Ex. 3. *To find the locus of the points of contact of tangent planes which make a given angle with the axis of z.*

We have
$$SkU(\phi\rho)=p,$$

$$Sk\phi\rho=pT\phi\rho,$$

or $$\frac{z^2}{c^4}=p^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right),$$

the equation of a cone whose axis is that of z and guiding curve an ellipse whose semi-axes are a^2, b^2.

The intersection of this surface with the ellipsoid is the locus required.

Ex. 4. *To find the locus of a point when the perpendicular from the centre on its polar plane is of constant length.*

Let π be the vector to the point, then

$S\rho\phi\pi=1$ is the equation of the polar plane (Art. 59),

and $T\dfrac{1}{\phi\pi}$ is the length of the perpendicular on it (Art. 58);

$$\therefore\ S(\phi\pi)^2 = -C^2, \text{ by the question.}$$

But since (44)

$$S\delta\phi\pi = S\pi\phi\delta,$$

if δ be $\phi\pi$,

$$S\phi\pi\phi\pi = S\pi\phi\phi\pi = S\pi\phi^2\pi;$$

$$\therefore\ S\pi\phi^2\pi = -C^2 \text{ is the equation required};$$

hence the Cartesian equation is (63. 6)

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} = C^2.$$

Ex. 5. *The sum of the squares of three conjugate semi-diameters is constant.*

Let α, β, γ be the semi-diameters; $\psi\alpha, \psi\beta, \psi\gamma$ are rectangular unit vectors (Art. 62).

Now $\quad \alpha = -(aiSi\psi\alpha + bjSj\psi\alpha + ckSk\psi\alpha)$ (63. 9);

$$\therefore\ (T\alpha)^2 = -\alpha^2 = a^2(Si\psi\alpha)^2 + b^2(Sj\psi\alpha)^2 + c^2(Sk\psi\alpha)^2,$$
$$(T\beta)^2 = a^2(Si\psi\beta)^2 + b^2(Sj\psi\beta)^2 + c^2(Sk\psi\beta)^2,$$
$$(T\gamma)^2 = a^2(Si\psi\gamma)^2 + b^2(Sj\psi\gamma)^2 + c^2(Sk\psi\gamma)^2:$$

adding, and observing that

$$(Si\psi\alpha)^2 + (Si\psi\beta)^2 + (Si\psi\gamma)^2 = 1 \quad (31.\ \text{Cor.}),$$

we get

$$(T\alpha)^2 + (T\beta)^2 + (T\gamma)^2 = a^2 + b^2 + c^2,$$
$$\text{i. e. } a'^2 + b'^2 + c'^2 = a^2 + b^2 + c^2.$$

Ex. 6. *The sum of the squares of the three perpendiculars from the centre on three tangent planes at right angles to one another is constant.*

We have

$$\rho = \phi^{-1}\phi\rho = a^2iSi\phi\rho + b^2jSj\phi\rho + c^2kSk\phi\rho \quad (63.\ 7),$$

and $\quad \phi\rho = -(iSi\phi\rho + jSj\phi\rho + kSk\phi\rho)$ (63. 2);

$$\therefore\ S\rho\phi\rho = 1 = a^2(Si\phi\rho)^2 + b^2(Sj\phi\rho)^2 + c^2(Sk\phi\rho)^2$$
$$= (T\phi\rho)^2\{a^2(SiU\phi\rho)^2 + b^2(SjU\phi\rho)^2 + c^2(SkU\phi\rho)^2\};$$

hence if ρ, ρ', ρ'' be three vectors so that $\phi\rho$, $\phi\rho'$, $\phi\rho''$ are at right angles to each other; that is, so that the tangent planes at their extremities are at right angles to one another (57. Cor.).

$$\begin{aligned}&\frac{1}{(T\phi\rho)^2}+\frac{1}{(T\phi\rho')^2}+\frac{1}{(T\phi\rho'')^2}\\ &=a^2\{(SiU\phi\rho)^2+(SiU\phi\rho')^2+(SiU\phi\rho'')^2\}\\ &+b^2\{(SjU\phi\rho)^2+\ldots\}+\ldots\\ &=a^2+b^2+c^2 \text{ (31. Cor.).}\end{aligned}$$

But $\frac{1}{(T\phi\rho)^2}$, &c. are the perpendiculars from the centre on the tangent planes at ρ, ρ', ρ'' (58). Hence the proposition.

Ex. 7. *The sum of the squares of the projections of three conjugate diameters on any of the principal axes is equal to the square of that axis.*

Let α, β, γ be conjugate semi-diameters; then, since

$$\alpha=-(aiSi\psi\alpha+bjSj\psi\alpha+ckSk\psi\alpha) \text{ (63. 9)},$$

$$Si\alpha=aSi\psi\alpha.$$

Similarly,

$$Si\beta=aSi\psi\beta,$$

$$Si\gamma=aSi\psi\gamma;$$

$$\begin{aligned}\therefore\ (Si\alpha)^2+(Si\beta)^2+(Si\gamma)^2&=a^2\{(Si\psi\alpha)^2+(Si\psi\beta)^2+(Si\psi\gamma)^2\}\\ &=a^2 \text{ (31. Cor.)},\end{aligned}$$

because $\psi\alpha$, $\psi\beta$, $\psi\gamma$ are at right angles to one another (62).

But $-Si\alpha$ is the projection of $T\alpha$ along the axis of x; and similarly of the others. Hence the proposition.

Ex. 8. *The sum of the reciprocals of the squares of the three perpendiculars from the centre on tangent planes at the extremities of conjugate diameters is constant.*

Let Oy_1, Oy_2, Oy_3 be the perpendiculars.

$$\begin{aligned}\frac{1}{Oy_1^{\,2}}&=-(\phi\alpha)^2 \text{ (58)}\\ &=\frac{(Si\alpha)^2}{a^4}+\frac{(Sj\alpha)^2}{b^4}+\frac{(Sk\alpha)^2}{c^4} \text{ (63. 3)};\end{aligned}$$

$$\frac{1}{Oy_2^{\,2}}=\frac{(Si\beta)^2}{a^4}+\frac{(Sj\beta)^2}{b^4}+\frac{(Sk\beta)^2}{c^4};$$

$$\frac{1}{Oy_3^{\,2}}=\frac{(Si\gamma)^2}{a^4}+\frac{(Sj\gamma)^2}{b^4}+\frac{(Sk\gamma)^2}{c^4};$$

$$\therefore\ \frac{1}{Oy_1^{\,2}}+\frac{1}{Oy_2^{\,2}}+\frac{1}{Oy_3^{\,2}}=\frac{1}{a^4}\left\{(Si\alpha)^2+(Si\beta)^2+(Si\gamma)^2\right\}+\&\text{c.}$$

$$=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\quad (\text{Ex. } 7).$$

Ex. 9. *If through a fixed point within an ellipsoid three chords be drawn mutually at right angles, the sum of the reciprocals of the products of their segments will be constant.*

Let θ be the vector to the given point; α, β, γ unit vectors parallel to three chords at right angles to each other.

Then $\theta+x\alpha=\rho$ gives

$$S(\theta+x\alpha)\,\phi\,(\theta+x\alpha)=1$$

a quadratic equation in x, the product of whose roots is

$$\frac{S\theta\phi\theta-1}{S\alpha\phi\alpha};$$

$\therefore$ the product of the reciprocals of the segments of the chord is

$$\frac{1}{x_{1\alpha}x_{2\alpha}}=\frac{S\alpha\phi\alpha}{S\theta\phi\theta-1}\cdot\frac{1}{(T\alpha)^2};$$

and the sum of the reciprocals of the products of the segments is

$$\frac{1}{S\theta\phi\theta-1}\cdot\left\{\frac{S\alpha\phi\alpha}{(T\alpha)^2}+\frac{S\beta\phi\beta}{(T\beta)^2}+\frac{S\gamma\phi\gamma}{(T\gamma)^2}\right\}.$$

Now since $\quad S\alpha\phi\alpha=\dfrac{(Si\alpha)^2}{a^2}+\dfrac{(Sj\alpha)^2}{b^2}+\dfrac{(Sk\alpha)^2}{c^2}\quad$ (63. 2, 3),

the sum of the reciprocals of the products

$$=\frac{1}{S\theta\phi\theta-1}\left[\frac{1}{a^2}\left\{(Si\alpha)^2+(Si\beta)^2+(Si\gamma)^2\right\}\right.$$

$$\left.+\frac{1}{b^2}\left\{(Sj\alpha)^2+(Sj\beta)^2+(Sj\gamma)^2\right\}\right.$$

$$+\frac{1}{c^2}\left\{(Ska)^2+\ldots\ldots\ldots\ldots\ldots\right\}\Bigg]$$

$$=\frac{1}{S\theta\phi\theta-1}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) \quad \text{(31. Cor.).}$$

COR. If θ be not constant, but $S\theta\phi\theta$ be so, i.e. if the given point be situated on an ellipsoid concentric with and similar to the given ellipsoid, the same is true.

Ex. 10. *If the poles lie in a plane parallel to yz, the polar planes cut the axis of x always in the same point.*

Let pi be the distance from the origin of the plane in which the poles lie, δ any line in that plane, then $\pi=pi+\delta$ is the vector to a pole, and

$$S\rho\phi\,(pi+\delta)=1 \quad (59)$$

the equation of the corresponding polar plane.

At the point where this plane cuts the axis of x,

$$\rho=xi\,;$$

$$\therefore\; Spxi\phi i+xSi\phi\delta=1.$$

Now δ is a vector in a plane perpendicular to ϕi,

$$\therefore\; Si\phi\delta=S\delta\phi i=0\,;$$

and $$Si\phi i=\text{constant}=n \text{ suppose}\,;$$

$$\therefore\; npx=1,$$

which shews that x is constant.

Ex. 11. *A, B and C are three similar and similarly situated ellipsoids; A and B are concentric, and C has its centre on the surface of B. To shew that the tangent plane to B at this point is parallel to the plane of intersection of A and C.*

Let a be the vector to the centre of C.

$$S\rho\phi\rho=a \text{ the equation of } A,$$

$$S\rho\phi\rho=b \;\ldots\ldots\ldots\ldots\ldots\ldots\; B,$$

$$S(\rho-a)\,\phi(\rho-a)=c\ldots\ldots\; C.$$

Now at the intersection of A and C, ρ is the same for both; therefore the equation of the plane of intersection is to be found by subtracting the one from the other.

It is therefore $\quad 2S\rho\phi\alpha = S\alpha\phi\alpha + a - c;$

and the equation of the tangent plane to B at the centre of C is

$$S\pi\phi\alpha = b;$$

$\therefore$ both planes are perpendicular to $\phi\alpha$, and are consequently parallel.

Ex. 12. *If through a given point chords be drawn to an ellipsoid, the intersections of pairs of tangent planes at their extremities all lie in a plane parallel to the tangent plane at the extremity of the diameter which passes through the point.*

Let α be the vector to the point; $\alpha + x_1\beta$, $\alpha + x_2\beta$, the vectors to the points of intersection with the ellipsoid of chords parallel to β; then

$$S\pi\phi(\alpha + x_1\beta) = 1,$$
$$S\pi\phi(\alpha + x_2\beta) = 1,$$

are the equations of the tangent planes at these points.

At the intersection of these planes π is the same for both; $\therefore$ subtracting we get

$$S\pi\phi\beta = 0,$$
$$S\pi\phi\alpha = 1.$$

The last equation is that of the line of intersection of the tangent planes; and that line is perpendicular to $\phi\alpha$, or (57. Cor.) parallel to the tangent plane at the extremity of the diameter which passes through the given point.

COR. $S\pi\phi\beta = 0$ shews that the line of intersection corresponding to any one chord is parallel to the tangent plane at the extremity of the diameter which is parallel to that chord.

Ex. 13. *Two similar and similarly situated ellipsoids are cut by a series of ellipsoids similar and similarly situated to the two*

given ones; and in such a manner that the planes of intersection are at right angles to one another. Shew that the centres of the cutting ellipsoids lie on another ellipsoid.

Let $$S\rho\phi\rho = 1 \quad\text{.............................}(1),$$
$$S(\rho - a)\,\phi\,(\rho - a) = C \quad\text{.....................}(2),$$
be the given ellipsoids;
$$S(\rho - \pi)\,\phi\,(\rho - \pi) = x \quad\text{..................}(3),$$
one of the cutting ellipsoids.

ϕ is the same for all because the ellipsoids are similar.

The plane of intersection of (1) and (3) is found by subtracting the equations; and is therefore
$$2S\rho\phi\pi = S\pi\phi\pi + 1 - x.$$

The plane of intersection of (2) and (3) is
$$2S\rho\,(\phi\pi - \phi a) = S\pi\phi\pi - Sa\phi a + C - x.$$

The former of these planes is perpendicular to $\phi\pi$ and the latter to $\phi\pi - \phi a$; and, since by the question, the former is perpendicular to the latter, $\phi\pi$ is perpendicular to $\phi\pi - \phi a$,
$$\therefore\ S\phi\pi\,(\phi\pi - \phi a) = 0,$$
the equation of the locus of the centres of the cutting ellipsoids.

This equation will be reduced to the requisite form by observing that
$$S\phi\pi\phi\pi = S\pi\phi\phi\pi = S\pi\phi^2\pi$$
$$S\phi\pi\phi a = Sa\phi^2\pi;$$
$$\therefore\ S(\pi - a)\,\phi^2\pi = 0,$$
the equation of an ellipsoid of which the semi-axes are proportional to
$$a^2,\ b^2,\ c^2 \quad (63.\ 6).$$

The Cartesian equation is
$$\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} - \left(\frac{xx'}{a^4} + \frac{yy'}{b^4} + \frac{zz'}{c^4}\right) = 0.$$

Ex. 14. *If a tangent plane be drawn to the inner of two similar concentric and similarly situated ellipsoids the point of contact is the centre of the elliptic section of the outer ellipsoid.*

Let $\quad S\rho\phi\rho = 1$ be the equation of the inner,

$a^2 S\rho\phi\rho = 1$ of the outer ellipsoid.

The tangent plane is $S\pi\phi\rho = 1$.

Now if σ be the vector to the elliptic section measured from the point of contact, $\pi = \rho + \sigma$ is a point in the outer ellipsoid;

$$\therefore \; a^2 S(\rho + \sigma)\,\phi(\rho + \sigma) = 1.$$

But
$$S\sigma\phi\rho = 0 \quad (57. \text{ Cor.});$$

$$\therefore \; a^2 + a^2 S\sigma\phi\sigma = 1,$$

$$\frac{a^2}{1 - a^2} S\sigma\phi\sigma = 1,$$

the equation of an ellipse of which the centre is the point of contact.

Ex. 15. *Find the equation of the curve described by a given point in a line of given length whose extremities move in fixed straight lines.*

First, let the straight lines lie in one plane.

Let unit vectors parallel to them be α, β.

Let the vectors of the extremities of the moving line be $x\alpha$, $y\beta$, and its length l. Then the condition is

$$(y\beta - x\alpha)^2 = -l^2,$$

or
$$x^2 + y^2 + 2xyS\alpha\beta = l^2 \quad (1).$$

The vector to a point which divides this line in the ratio $e : 1$ is

$$\rho = x\alpha + e(y\beta - x\alpha)$$
$$= x\alpha(1 - e) + ey\beta;$$

$$\therefore \; S\alpha\rho = -(1 - e)x + eyS\alpha\beta,$$
$$S\beta\rho = (1 - e)xS\alpha\beta - ey;$$

whence $x = \frac{Sa\rho + Sa\beta S\beta\rho}{(1-e)(S^2a\beta-1)}$, $y = \frac{S\beta\rho + Sa\beta Sa\rho}{e(S^2a\beta-1)}$,

which values being substituted in equation (1) give the required equation, viz.:

$$\frac{(Sa\rho + Sa\beta S\beta\rho)^2}{(1-e)^2} + \frac{(S\beta\rho + Sa\beta Sa\rho)^2}{e^2}$$
$$+ 2\frac{Sa\beta}{e(1-e)}(Sa\rho + Sa\beta S\beta\rho)(S\beta\rho + Sa\beta Sa\rho)$$
$$= l^2(S^2a\beta - 1)^2.$$

But ρ is subject to the additional condition (31. 2. Cor. 2) $S.a\beta\rho = 0$; and the locus is a plane ellipse.

When the given straight lines are at right angles to one another, the equation is much simplified, for

$$Sa\beta = 0;$$

and our equations are

$$x^2 + y^2 = l^2,$$
$$Sa\rho = -(1-e)x, \quad S\beta\rho = -ey;$$

whence $$\frac{(Sa\rho)^2}{(1-e)^2} + \frac{(S\beta\rho)^2}{e^2} = l^2,$$

an ellipse of which the semi-axes are le and $l(1-e)$.

Generally, if the given lines do not meet, let the origin be chosen midway along the line perpendicular to both; then we have

$$\{\gamma + xa - (-\gamma + y\beta)\}^2 = -l^2,$$

γ and $-\gamma$ being the vectors perpendicular to the lines,

$$\rho = (\gamma + xa)(1-e) + e(-\gamma + y\beta).$$

The first gives

$$4\gamma^2 + (xa - y\beta)^2 = -l^2;$$

and the second gives, as in the simpler case above,

$$Sa\rho = -(1-e)x + eySa\beta,$$
$$S\beta\rho = (1-e)xSa\beta - ey.$$

Hence the elimination of x and y again leads to the equation of an ellipsoid, the only difference being that l^2 is diminished by the square of the shortest distance between the lines; i.e. the axes are less than in the former case.

In the extreme case, where $l = 2T\gamma$, the equation cannot be satisfied except by

$$x = 0, \quad y = 0,$$

(i. e. the locus is reduced to a single point), unless indeed we have

$$\alpha = \pm \beta,$$

for then
$$x = \pm y,$$

and the locus is a straight line parallel to each of the preceding lines.

65. *The cone.*

1. To find the equation of a cone of revolution whose vertex is the origin O.

Let α be a unit vector along the axis OA,

ρ the vector to a point P on the surface of the cone;

then
$$S\alpha\rho = -T\rho \cos\theta,$$

θ being the angle POA.

But this angle is constant,

$$\therefore\ S^2\alpha\rho = c^2\rho^2 \text{ is the equation required.}$$

2. The equation of a cone which has circular sections, but which is not necessarily a cone of revolution, is thus found.

Take the vertex as the origin, and let one of the circular sections be the intersection of the plane

$$S\alpha\rho = -a^2 \qquad\text{.............................}(1)$$

with the sphere
$$\rho^2 = S\beta\rho \qquad\text{.............................}(2).$$

Since these are scalar equations we may multiply them together; and thus obtain at all the points of the circular section

$$a^2\rho^2 + S\alpha\rho S\beta\rho = 0 \qquad\text{......................}(3).$$

Now if $x\rho$ or ρ' be written in place of ρ, the equation is not changed, since ρ occurs twice on each side. It is therefore the required equation of the cone.

COR. 1. Every section by a plane parallel to $S\alpha\rho = -a^2$ is a circle.

For the equation of a plane parallel to

$$S\alpha\rho = -a^2$$

is

$$S\alpha\rho = -aa^2,$$

which being substituted in the equation of the cone gives

$$\rho^2 = aS\beta\rho,$$

the equation of a circle.

COR. 2. The plane $$S\beta\rho = -b\beta^2 \qquad (4)$$
also gives a circle whose equation is

$$a^2\rho^2 = b\beta^2 S\alpha\rho \qquad (5).$$

These two equations give the *subcontrary* sections.

To deduce the relation between the two sections; let O be the vertex of the cone, OAB the plane through α, β; AB the line in which the section cuts this plane, AD that in which the subcontrary section cuts it;

$$OA = \rho, \quad OB = \rho', \quad OD = x\rho'.$$

We have, by (5), $$x\rho'^2 = \frac{b\beta^2}{a^2} S\alpha\rho'$$

$$= -b\beta^2, \text{ by (1)},$$

$$= S\beta\rho, \text{ by (4)},$$

$$= \rho^2, \text{ by (2)};$$

i. e. $$OB \,.\, OD = OA^2,$$

and the triangles OAB, OAD are similar, or AD cuts OA at the same angle that AB cuts OB.

66. If $$\phi\rho = 2a^2\rho + \alpha S\beta\rho + \beta S\alpha\rho,$$
the equation of the cone is reduced to

$$S\rho\phi\rho = 0.$$

It is evident that all the properties of $\phi\rho$, Art. 44, are applicable here.

As in Art. 57, the equation of the tangent plane is

$$S\pi\phi\rho = 0.$$

67. EXAMPLES.

Ex. 1. *Tangent planes are drawn to an ellipsoid from a given external point, to find the cone which has its vertex at the origin, and which passes through all the points of contact of the tangent planes with the ellipsoid.*

Let α be the vector to the external point, ρ a point in the ellipsoid where a tangent plane through α touches it.

Then the equation of the ellipsoid is

$$S\rho\phi\rho = 1,$$

and the equation of the tangent plane

$$S\alpha\phi\rho = 1, \quad \text{i. e. } S\rho\phi\alpha = 1.$$

The equation

$$S\rho\phi\rho = (S\rho\phi\alpha)^2,$$

or $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \left(\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2}\right)^2,$$

represents a surface passing through the points of contact; and is the cone required.

Ex. 2. *Of a system of three rectangular vectors two are confined to given planes, to find the surface traced out by the third.*

Let π, ρ, σ be the three vectors, of which two are confined to given planes whose equations are

$$S\alpha\pi = 0, \quad S\beta\rho = 0,$$

to find the locus of σ.

Since the vectors are at right angles, we have

$$S\pi\rho = 0, \quad S\pi\sigma = 0, \quad S\sigma\rho = 0,$$

and we have five equations from which to eliminate π and ρ.

Since $$S\alpha\pi = 0, \quad S\sigma\pi = 0,$$
π is at right angles to both α and σ, and therefore to the plane $\alpha\sigma$; or

$$\pi = xV\alpha\sigma.$$

Since $$S\beta\rho = 0, \quad S\sigma\rho = 0,$$
ρ is at right angles to the plane $\beta\sigma$; therefore
$$\rho = yV\beta\sigma,$$
and $$\pi\rho = xy\,V\alpha\sigma\,V\beta\sigma.$$
Now $$S\pi\rho = 0,$$
therefore $$S\,.\,V\alpha\sigma\,V\beta\sigma = 0,$$
or $$S\,(\alpha\sigma - S\alpha\sigma)\,(\beta\sigma - S\beta\sigma) = 0,$$
or $$\sigma^2 S\alpha\beta - S\alpha\sigma S\beta\sigma = 0,$$
the equation of a cone of the second order, which has circular sections (65. 2).

Cor. The circular sections are parallel to the two planes to which the two vectors are confined.

Ex. 3. *The equation* $\rho = t^2\alpha + u^2\beta + (t+u)^2\gamma = 0$ *is that of a cone of the second order touched by each of the three planes through OAB, OBC, OCA; and the section ABC through the extremities of* α, β, γ *is an ellipse touched at their middle points by AB, BC, CA.*

1. If the surface be referred to oblique co-ordinates parallel to α, β, γ respectively, we shall have
$$\rho = x\alpha + y\beta + z\gamma,$$
therefore $$x = t^2, \quad y = u^2, \quad z = (t+u)^2,$$
or $$z = (\sqrt{x} + \sqrt{y})^2 = x + y + 2\sqrt{xy},$$
which gives $$(z - x - y)^2 = 4xy,$$
a cone of the second order.

2. If $t = -u$, the equation becomes
$$\rho = t^2(\alpha + \beta),$$
the equation of a straight line bisecting the base AB, which since it satisfies the equation relative to t, shews that this line coincides with the cone in all its length; i.e. the cone is touched in this line by the plane OAB.

Similarly, by putting $t = 0$, $u = 0$ respectively, we can shew that the cone is touched by the plane BOC, COA in the lines which bisect AC, CA.

3. Restricting ourselves to the plane ABC, we have the section of a cone of the second order enclosed by the triangle ABC, which triangle is itself the section of three planes each of which touches the cone.

Ex. 4. *The equation* $\rho = a\alpha + b\beta + c\gamma$ *with the condition* $ab + bc + ca = 0$ *is a cone of the second order, and the lines* OA, OB, OC *coincide throughout their length with the surface.*

1. It is evident that the equation gives

$$xy + yz + zx = 0.$$

2. That if $b = 0$, $c = 0$, the question is satisfied by

$$\rho = a\alpha,$$

whatever be a, therefore &c.

Ex. 5. *Find the locus of a point, the sum of the squares of whose distances from a number of given planes is constant.*

Let $S\delta_1\rho_1 = C_1$, $S\delta_2\rho_2 = C_2$, &c. be the equations of the given planes, ρ the vector to the point under consideration; then $x_1\delta_1$, $x_2\delta_2$, &c. will be the perpendiculars on the planes from the point; provided

$$\rho + x_1\delta_1 = \rho_1, \quad \rho + x_2\delta_2 = \rho_2, \text{ \&c.};$$

therefore $$S\delta_1(\rho + x_1\delta_1) = C_1, \text{ \&c.}$$

and $$x_1\delta_1^2 = C_1 - S\delta_1\rho, \text{ \&c.},$$

$$x_1^2\delta_1^4 = (C_1 - S\delta_1\rho)^2;$$

i.e. the square of the line perpendicular to the first plane from the given point

$$= \left(\frac{C_1 - S\delta_1\rho}{T\delta_1}\right)^2,$$

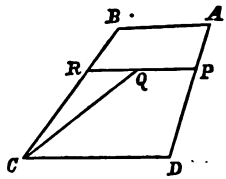
and, by the question,

$$\left(\frac{C_1 - S\delta_1\rho}{T\delta_1}\right)^2 + \left(\frac{C_2 - S\delta_2\rho}{T\delta_2}\right)^2 + \text{\&c. is constant.}$$

The locus is therefore a surface of the second order.

10—2

Ex. 6. *The lines which divide proportionally the pairs of opposite sides of a gauche quadrilateral, are the generating lines of a hyperbolic paraboloid.*



Let $ABCD$ be the quadrilateral.

AD, BC are divided proportionally in P and R.

Let $CA = \alpha$, $CB = \beta$, $CD = \gamma$;

$$CR = m\beta, \quad DP = mDA;$$

i.e. $$CP - \gamma = m(\alpha - \gamma);$$

therefore $$RP = CP - CR = \gamma + m(\alpha - \gamma) - m\beta,$$

$$\rho = CQ = CR + pRP$$
$$= m\beta + p\{\gamma + m(\alpha - \gamma) - m\beta\}$$
$$= x\alpha + y\beta + z\gamma, \text{ say};$$

therefore $$x = pm, \quad y = m - pm, \quad z = p(1 - m);$$

therefore $$m = x + y, \quad p = \frac{x}{x+y},$$

$$z = \frac{x}{x+y} - x,$$

or $$(x+z)(x+y) = x,$$

the equation referred to oblique co-ordinates parallel to α, β, γ.

Pascal's Hexagram.

68. Let O be the origin, OA, OB, OC, OD, OE five given vectors lying on the surface of a cone, and terminated in a plane section of the cone $ABCDEF$, not passing through O; OX *any* vector lying on the same surface.

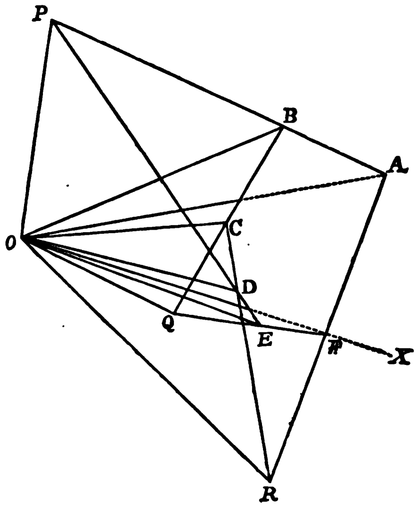
Let $OA = \alpha$, $OB = \beta$, $OC = \gamma$, $OD = \delta$, $OE = \epsilon$, $OX = \rho$.

The equation

$$S.\, V(V\alpha\beta V\delta\epsilon)\, V(V\beta\gamma V\epsilon\rho)\, V(V\gamma\delta V\rho\alpha) = 0 \quad \text{.......(1)}$$

is the equation of a cone of the second order whose vertex is O *and vector* ρ along the surface. For

1. It is a cone whose vertex is O because it is not altered by writing $x\rho$ for ρ. Also it is of the second order in ρ, since ρ occurs in it twice and twice only.



2. All the vectors OA, OB, OC, OD, OE lie on its surface.

This we shall prove by shewing that if ρ coincide with any one of them the equation (1) is satisfied.

If ρ coincide with α, the last term of the left-hand side of the equation, viz. $V\rho\alpha$, becomes $V\alpha\alpha = V\alpha^2 = 0$, and the equation is satisfied.

If ρ coincide with β, the left-hand side of the equation becomes

$$S.V(V\alpha\beta V\delta\epsilon)\,V(V\beta\gamma V\epsilon\beta)\,V(V\gamma\delta V\beta\alpha)\,\dots\dots\dots(2).$$

Now $V(V\beta\gamma V\epsilon\beta) = -V(V\epsilon\beta V\beta\gamma)$, (22. 2), is a vector parallel to β (31. 3), call it $m\beta$; and

$$V.\{V(V\alpha\beta V\delta\epsilon)\,V(V\gamma\delta V\beta\alpha)\} = V.\{V(V\alpha\beta V\delta\epsilon)\,V(V\alpha\beta V\gamma\delta)\},\ (22.\ 2),$$
$$= \text{a multiple of } V\alpha\beta,\ (31.\ 3),$$
$$= nV\alpha\beta, \text{ say.}$$

Hence the product of the first and third vectors in expression (2) becomes

$$\text{scalar} + nV\alpha\beta,$$

and the second is $m\beta$; therefore expression (2) becomes, by 31. 2,

$$\begin{aligned} &S.(\text{scalar} + nV\alpha\beta)\,m\beta \\ &= Smn\beta V\alpha\beta \\ &= 0, \end{aligned}$$

because $V\alpha\beta$ is a vector perpendicular to β.

Equation (1) is therefore satisfied when ρ coincides with β.

If ρ coincide with γ both the second and third vectors are parallel to β (31. 3); therefore their product is a scalar, and equation (1) is satisfied.

The other cases are but repetitions of these.

Hence equation (1) is satisfied if ρ coincide with any one of the five vectors α, β, γ, δ, ϵ; i.e. OA, OB, OC, OD, OE are vectors on the surface of the cone.

3. Let F be the point in which OX cuts the plane $ABCDE$; then $ABCDEF$ are the angular points of a hexagon inscribed in a conic section.

4. Let the planes OAB, ODE intersect in OP; OBC, OEF in OQ; OCD, OFA in OR; then

$$\begin{aligned} &V.\,V\alpha\beta V\delta\epsilon = mOP, \ (31.\ 4), \\ &V.\,V\beta\gamma V\epsilon\rho = nOQ, \\ &V.\,V\gamma\delta V\rho\alpha = pOR; \end{aligned}$$

therefore

$$S.\,V(V\alpha\beta V\delta\epsilon)\,V(V\beta\gamma V\epsilon\rho)\,V(V\gamma\delta V\rho\alpha) = mnpS(OP.OQ.OR);$$

hence equation (1) gives

$$S(OP.OQ.OR) = 0,$$

or (31. 2. Cor. 2) OP, OQ, OR are in the same plane.

Hence PQR, the intersection of this plane with the plane $ABCDEF$ is a straight line. But P is the point of intersection of AB, ED, &c.

Therefore, the opposite sides (1st and 4th, 2nd and 5th, 3rd and 6th) of a hexagon inscribed in a conic section being produced meet in the same straight line.

COR. It is evident that the demonstration applies to any six points in the conic, whether the lines which join them form a hexagon or not.

ADDITIONAL EXAMPLES TO CHAP. VIII.

1. Find the locus of a point, the ratio of whose distances from two given straight lines is constant.

2. Find the locus of a point the square of whose distance from a given line is proportional to its distance from a given plane.

3. Prove that the locus of the foot of the perpendicular from the centre on the tangent plane of an ellipsoid is

$$(ax)^2 + (by)^2 + (cz)^2 = (x^2 + y^2 + z^2)^2.$$

4. The sum of the squares of the reciprocals of any three radii at right angles to one another is constant.

5. If Oy_1, Oy_2, Oy_3 be perpendiculars from the centre on tangent planes at the extremities of conjugate diameters, and if Q_1, Q_2, Q_3 be the points where they meet the ellipsoid; then

$$\frac{1}{OY_1^2 . OQ_1^2} + \frac{1}{OY_2^2 . OQ_2^2} + \frac{1}{OY_3^2 . OQ_3^2} = \frac{1}{a^4} + \frac{1}{b^4} + \frac{1}{c^4}.$$

6. If tangent planes to an ellipsoid be drawn from points in a plane parallel to that of xy, the curves which contain all the points of contact will lie in planes which all cut the axis of z in the same point.

7. Two similar and similarly situated ellipsoids intersect in a plane curve whose plane is conjugate to the line which joins the centres of the ellipsoids.

8. If points be taken in conjugate semi-diameters produced, at distances from the centre equal to p times those semi-diameters respectively; the sum of the squares of the reciprocals of the

perpendiculars from the centre on their polar planes is equal to p^2 times the sum of the squares of the perpendiculars from the centre on tangent planes at the extremities of those diameters.

9. If P be a point on the surface of an ellipsoid, PA, PB, PC any three chords at right angles to each other, the plane ABC will pass through a fixed point, which is in the normal to the ellipsoid at P; and distant from P by

$$\frac{\frac{2}{p}}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}},$$

where p is the perpendicular from the centre on the tangent plane at P.

10. Find the equation of the cone which has its vertex in a given point, and which touches and envelopes a given ellipsoid.

CHAPTER IX.

FORMULÆ AND THEIR APPLICATION.

69. Products of two or more vectors.

1. *Two vectors.* The relations which exist between the scalars and vectors of the product of two vectors have already been exhibited in Art. 22. We simply extract them:

(a) $S\alpha\beta = S\beta\alpha.$ (b) $V\alpha\beta = -V\beta\alpha.$

(c) $\alpha\beta + \beta\alpha = 2S\alpha\beta.$ (d) $\alpha\beta - \beta\alpha = 2V\alpha\beta.$

These we shall quote as formulæ (1).

2. We may here add a single conclusion for quaternion products.

Any quaternion, such as $\alpha\beta$, may be written as the sum of a scalar and a vector. If therefore q and r be quaternions, we may write

$$q = Sq + Vq,$$

$$r = Sr + Vr;$$

therefore $$qr = SqSr + SqVr + SrVq + VqVr,$$

and $$S \, . \, qr = SqSr + S \, . \, VqVr,$$

$$V \, . \, qr = SqVr + SrVq + V \, . \, VqVr,$$

where $S \, . \, VqVr$ is the scalar part, and $V \, . \, VqVr$ the vector part of the product of the two vectors Vq, Vr.

If now we transpose q and r, and apply (a) and (b) of formulæ 1, we get

$$\left. \begin{aligned} S \, . \, qr &= S \, . \, rq \\ V \, . \, qr + V \, . \, rq &= 2\,(SqVr + SrVq) \end{aligned} \right\} \quad \ldots\ldots\ldots\ldots\ldots (2).$$

3. *Three vectors.* By observing that $S.\gamma S\alpha\beta$ is simply the scalar of a vector, and is consequently zero, we may insert or omit such an expression at pleasure. By bearing this in mind the reader will readily apprehend the demonstrations which follow, even in cases where we have studied brevity.

$$\begin{aligned} S.\alpha\beta\gamma &= S.(S\alpha\beta + V\alpha\beta)\gamma \\ &= S.\gamma V\alpha\beta, \text{ (by 1. } a), \\ &= S.\gamma(S\alpha\beta + V\alpha\beta) \\ &= S.\gamma\alpha\beta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3). \end{aligned}$$

Again,

$$\begin{aligned} S.\alpha\beta\gamma &= S.\alpha(S\beta\gamma + V\beta\gamma) \\ &= S(V\beta\gamma.\alpha), \text{ (by 1. } a), \\ &= S(S\beta\gamma + V\beta\gamma)\alpha \\ &= S.\beta\gamma\alpha \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3). \end{aligned}$$

The formulæ marked (3) shew that a change of order amongst three vectors produces no change in the scalar of their product, provided the cyclical order remain unchanged.

This conclusion might have been obtained by a different process, thus:

In (2) let $q = \alpha\beta$, $r = \gamma$, there results at once

$$S.\alpha\beta\gamma = S.\gamma\alpha\beta.$$

Again in (2) let $q = \gamma\alpha$, $r = \beta$, there results

$$S.\gamma\alpha\beta = S.\beta\gamma\alpha.$$

We have therefore, as before,

$$S.\alpha\beta\gamma = S.\gamma\alpha\beta = S.\beta\gamma\alpha \ldots\ldots\ldots\ldots\ldots (3).$$

4.

$$\begin{aligned} S.\alpha\beta\gamma &= S.\alpha V\beta\gamma \\ &= -S.\alpha V\gamma\beta, \text{ (by 1. } b), \\ &= -S.\alpha\gamma\beta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4). \end{aligned}$$

Similarly

$$S.\alpha\beta\gamma = -S.\beta\alpha\gamma \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

or a cyclical change of order amongst three vectors changes the sign of the scalar of their product.

5. It has already been seen (Art. 31. 1) that $-S.\alpha\beta\gamma$ is the volume of the parallelepiped of which the three edges which terminate in the point O are the lines OA, OB, OC whose vectors are α, β, γ respectively.

We may express this volume in the form of a determinant, thus:

Let α, β, γ be replaced by

$$xi+yj+zk,\quad x'i+y'j+z'k,\quad x''i+y''j+z''k \text{ (Art. 31. 5)};$$

x, y, z being the rectangular co-ordinates of A, x', y', z' those of B, x'', y'', z'' those of C, measured from O as the origin; then

$$\begin{aligned} S.\alpha\beta\gamma = S.&(xi+yj+zk)\\ &\times(x'i+y'j+z'k)\\ &\times(x''i+y''j+z''k). \end{aligned}$$

Now if we observe first that the scalar part of this product is confined to those terms in which all the three vectors i, j, k appear; and secondly that the sign of any term in the product will by formulæ (3) and (4) be $-$ or $+$ according as cyclical order is or is not retained, we perceive that we have the exact conditions which apply to a determinant; therefore

$$S.\alpha\beta\gamma = -\begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ x'', & y'', & z'' \end{vmatrix} \quad \text{..................(5).}$$

The volume of the pyramid $OABC$ is one-sixth of the above.

Note relative to the sign of the scalar.

Since $ijk=-1$ (19), it is clear that if OA, OB, OC assume the positions of Ox, Oy, Oz in the figure of Art. 16, $S(OA.OB.OC)$ will have a *minus* sign, whilst the order of the letters A, B, C is right-handed as seen from O.

If now we take any pyramid whatever $OABC$, of which the vertex is O, and assume that $S(OA.OB.OC)$ (which, being proportional to the volume of the pyramid, we may designate $OABC$), is negative when the order of the letters A, B, C is right-handed

as seen from O, we shall find the following general law of signs to hold good whatever be the vertex; viz. *the sign of the scalar is minus or plus according as the order in it of the angles of the base of the pyramid is right-handed or left-handed as seen from the vertex.*

For example,
$$\begin{aligned} CABO &= S(CA \, . \, CB \, . \, CO) \\ &= S(\alpha - \gamma)(\beta - \gamma)(-\gamma) \\ &= - S\alpha\beta\gamma \\ &= - OABC, \end{aligned}$$

which is *plus* because $OABC$ is *minus*, and the order of the letters A, B, O as seen from C is left-handed.

6.
$$\begin{aligned} V \, . \, \alpha\beta\gamma &= V \, . \, \alpha \, (S\beta\gamma + V\beta\gamma) \\ &= \alpha S\beta\gamma + V \, . \, \alpha V\beta\gamma; \\ V \, . \, \gamma\beta\alpha &= V \, . \, (S\gamma\beta + V\gamma\beta) \, \alpha \\ &= \alpha S\beta\gamma - V \, . \, \alpha V\gamma\beta, \ (1. \ b), \\ &= \alpha S\beta\gamma + V \, . \, \alpha V\beta\gamma, \ (1. \ b), \\ &= V \, . \, \alpha\beta\gamma \ \ldots\ldots\ldots\ldots (6). \end{aligned}$$

7.
$$\begin{aligned} V \, . \, \alpha\beta\gamma &= V \, . \, (S\alpha\beta + V\alpha\beta) \, \gamma \\ &= \gamma S\alpha\beta - V \, . \, \gamma V\alpha\beta; \\ V \, . \, \gamma\alpha\beta &= V \, . \, \gamma \, (S\alpha\beta + V\alpha\beta) \\ &= \gamma S\alpha\beta + V \, . \, \gamma V\alpha\beta; \end{aligned}$$

therefore
$$V \, . \, \alpha\beta\gamma + V \, . \, \gamma\alpha\beta = 2\gamma S\alpha\beta \ \ldots\ldots\ldots\ldots (7).$$

8.
$$\begin{aligned} 2V \, . \, \alpha V\beta\gamma &= V \, . \, \alpha \, (\beta\gamma - \gamma\beta), \ (1. \ d), \\ &= V \, . \, \alpha\beta\gamma + V \, . \, \gamma\alpha\beta - (V \, . \, \alpha\gamma\beta + V \, . \, \gamma\alpha\beta) \\ &= V(\alpha\beta\gamma + \beta\alpha\gamma) - V(\alpha\gamma\beta + \gamma\alpha\beta), \ (\text{by } 6), \\ &= V \, . \, (\alpha\beta + \beta\alpha) \, \gamma - V \, . \, (\alpha\gamma + \gamma\alpha) \, \beta \\ &= 2\gamma S\alpha\beta - 2\beta S\alpha\gamma, \ (1. \ c); \end{aligned}$$

therefore
$$V \, . \, \alpha V\beta\gamma = \gamma S\alpha\beta - \beta S\alpha\gamma \ \ldots\ldots\ldots\ldots (8).$$

9. We have, by (8),

$$V.\alpha V\beta\gamma = \gamma S\alpha\beta - \beta S\alpha\gamma,$$
$$V.\beta V\gamma\alpha = \alpha S\beta\gamma - \gamma S\alpha\beta,$$
$$V.\gamma V\alpha\beta = \beta S\alpha\gamma - \alpha S\beta\gamma;$$

therefore, by addition,

$$V.(\alpha V\beta\gamma + \beta V\gamma\alpha + \gamma V\alpha\beta) = 0 \quad\ldots\ldots\ldots\ldots\ldots(9),$$

10. $$V.\alpha\beta\gamma = V.\alpha(S\beta\gamma + V\beta\gamma)$$
$$= \alpha S\beta\gamma + V.\alpha V\beta\gamma,$$

which, by (8), $$= \alpha S\beta\gamma - \beta S\alpha\gamma + \gamma S\alpha\beta \quad\ldots\ldots\ldots\ldots(10).$$

Another proof of this important formula is found in the identity

$$\frac{1}{2}(\alpha\beta\gamma + \gamma\beta\alpha) = \frac{1}{2}\alpha(\beta\gamma + \gamma\beta) - \frac{1}{2}\beta(\alpha\gamma + \gamma\alpha) + \frac{1}{2}\gamma(\alpha\beta + \beta\alpha),$$

which, by (4) and (6), is the theorem itself.

11. If in (8) we write $V\alpha\beta$ in place of α, we get

$$V.V\alpha\beta V\beta\gamma = \gamma S(V\alpha\beta.\beta) - \beta S(V\alpha\beta.\gamma)$$
$$= \gamma S.\alpha\beta\beta - \beta S\alpha\beta\gamma$$
$$= -\beta S.\alpha\beta\gamma \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11).$$

12. *Four vectors.* If in (8) we write $V\alpha\delta$ in place of α, we obtain

$$V(V\alpha\delta V\beta\gamma) = \gamma S.\alpha\delta\beta - \beta S.\alpha\delta\gamma \quad\ldots\ldots\ldots\ldots(12).$$

13. By (12) we have

$$V(V\beta\gamma V\alpha\delta) = \delta S.\beta\gamma\alpha - \alpha S.\beta\gamma\delta.$$

But $$V(V\beta\gamma V\alpha\delta) = -V(V\alpha\delta V\beta\gamma).$$

Hence, by adding the above result to (12), we get

$$\delta S.\beta\gamma\alpha - \alpha S.\beta\gamma\delta + \gamma S.\alpha\delta\beta - \beta S.\alpha\delta\gamma = 0,$$

which, by (3) and (4), if we adopt alphabetical order, may be written

$$\alpha S.\beta\gamma\delta - \beta S.\alpha\gamma\delta + \gamma S.\alpha\beta\delta - \delta S.\alpha\beta\gamma = 0 \quad\ldots\ldots(13),$$

or. $$\delta S.\alpha\beta\gamma = \alpha S.\beta\gamma\delta - \beta S.\alpha\gamma\delta + \gamma S.\alpha\beta\delta \quad\ldots\ldots\ldots\ldots(13).$$

or, again, if we adopt cyclical order,

$$\alpha S.\beta\gamma\delta - \delta S.\alpha\beta\gamma + \gamma S.\delta\alpha\beta - \beta S.\gamma\delta\alpha,$$

or, finally, $\delta S.\alpha\beta\gamma = \alpha S.\beta\gamma\delta - \beta S.\gamma\delta\alpha + \gamma S.\delta\alpha\beta$.........(13).

This equation expresses a vector in terms of three other vectors. The following equation expresses it in terms of the vectors which result from their products two and two.

14. $V(\gamma\delta\alpha\beta)$ may be written, first as $V(\gamma.\delta\alpha\beta)$, and secondly as $V(\gamma\delta.\alpha\beta)$, and the results compared. These forms give respectively

$$\begin{aligned}
V(\gamma.\delta\alpha\beta) &= V.\gamma(S.\delta\alpha\beta + V.\delta\alpha\beta)\\
&= \gamma S.\alpha\beta\delta + V.\gamma(\delta S\alpha\beta - \alpha S\delta\beta + \beta S\delta\alpha), \text{ by (3) and (10)},\\
&= \gamma S.\alpha\beta\delta + V\gamma\delta S\alpha\beta - V\gamma\alpha S\delta\beta + V\gamma\beta S\delta\alpha;\\
V(\gamma\delta.\alpha\beta) &= V.(S\gamma\delta + V\gamma\delta)(S\alpha\beta + V\alpha\beta)\\
&= V\alpha\beta S\gamma\delta + V\gamma\delta S\alpha\beta + V.V\gamma\delta V\alpha\beta\\
&= V\alpha\beta S\gamma\delta + V\gamma\delta S\alpha\beta - V.V\alpha\beta V\gamma\delta\\
&= V\alpha\beta S\gamma\delta + V\gamma\delta S\alpha\beta - \delta S.\alpha\beta\gamma + \gamma S.\alpha\beta\delta, \text{ by (12)}.
\end{aligned}$$

The two expressions being equated, and the common terms deleted, there results

$$\delta S.\alpha\beta\gamma = V\alpha\beta S\gamma\delta + V\beta\gamma S\alpha\delta + V\gamma\alpha S\beta\delta \text{.........(14)}.$$

15. $$\begin{aligned}
S.\alpha\beta\gamma\delta &= S.(S.\alpha\beta\gamma + V.\alpha\beta\gamma)\delta\\
&= S.(V.\alpha\beta\gamma)\delta\\
&= S.(\alpha S\beta\gamma - \beta S\alpha\gamma + \gamma S\alpha\beta)\delta, \text{ by (10)},\\
&= S\alpha\beta S\gamma\delta - S\alpha\gamma S\beta\delta + S\alpha\delta S\beta\gamma \text{ ....(15)}.
\end{aligned}$$

16. $$\begin{aligned}
S(V\alpha\beta V\gamma\delta) &= S.(\alpha\beta - S\alpha\beta)(\gamma\delta - S\gamma\delta)\\
&= S.\alpha\beta\gamma\delta - S\alpha\beta S\gamma\delta\\
&= S\alpha\delta S\beta\gamma - S\alpha\gamma S\beta\delta, \text{ by (15) ....(16)}.
\end{aligned}$$

17. $$\begin{aligned}
S.\alpha\beta\gamma\delta &= S.(V\alpha\beta\gamma)\delta\\
&= S.\delta V\alpha\beta\gamma\\
&= S.\delta\alpha\beta\gamma \text{......................(17)}.
\end{aligned}$$

18. *Five vectors.* As we do not purpose to exhibit any applications of the relations which exist among five or more vectors, we shall confine ourselves to simply writing down the two following expressions.

$$S \,.\, \alpha\beta\gamma\delta\epsilon = - S \,.\, \epsilon\delta\gamma\beta\alpha,$$

$$V \,.\, \alpha\beta\gamma\delta\epsilon = V \,.\, \epsilon\delta\gamma\beta\alpha \text{.....................} (18).$$

70. Many of these formulæ might have been proved differently, and some of them more directly, by assuming for instance that α, β, γ are not in the same plane. In this case *any* other vector δ may be expressed in terms of α, β, γ, by the equation

$$\delta = x\alpha + y\beta + z\gamma, \ (31.\ 5);$$

therefore
$$S \,.\, \beta\gamma\delta = xS \,.\, \beta\gamma\alpha = xS \,.\, \alpha\beta\gamma, \ (3),$$

$$S \,.\, \gamma\delta\alpha = yS \,.\, \gamma\beta\alpha = -yS \,.\, \alpha\beta\gamma, \ (4),$$

$$S \,.\, \delta\alpha\beta = zS \,.\, \gamma\alpha\beta = zS \,.\, \alpha\beta\gamma, \ (3);$$

therefore
$$\delta S \,.\, \alpha\beta\gamma = x\alpha S \,.\, \alpha\beta\gamma + y\beta S \,.\, \alpha\beta\gamma + z\gamma S \,.\, \alpha\beta\gamma$$

$$= \alpha S \,.\, \beta\gamma\delta - \beta S \,.\, \gamma\delta\alpha + \gamma S \,.\, \delta\alpha\beta$$

which is formula 13.

71. EXAMPLES.

Ex. 1. *To express the relation between the sides of a spherical triangle and the angles opposite to them.*

Retaining the notation and figure of Ex. 2, Art. 29, we shall have

$$V\alpha\beta V\beta\gamma = \gamma' \sin c \,.\, \alpha' \sin a,$$

where γ', α' are unit vectors perpendicular respectively to the planes OAB, OBC.

Therefore $\quad V \,.\, V\alpha\beta V\beta\gamma = \sin c \sin a \,.\, \beta \sin B.$

Also $\quad -\beta S \,.\, \alpha\beta\gamma = \beta \sin c \sin \phi, \ (31.\ 1),$

where ϕ is the angle between OC and the plane OAB.

Now these results are equal (formula 11), therefore

$$\sin \phi = \sin a \sin B.$$

Similarly $\sin\phi = \sin b \sin A$;

therefore $\sin a \sin B = \sin b \sin A$,

or $\sin a : \sin b :: \sin A : \sin B$.

Ex. 2. *To find the condition that the perpendiculars from the angles of a tetrahedron on the opposite faces shall intersect one another.*

Let OA, OB, OC be the edges of the tetrahedron (Fig. of Art. 31), α, β, γ the corresponding vectors.

Vector perpendiculars from A and B on the opposite faces are $V\beta\gamma$, $V\gamma\alpha$ respectively (22. 8). If these perpendiculars intersect in G, the three points A, B, G will be in one plane, whence

$$S.(\beta-\alpha)\,V\beta\gamma V\gamma\alpha = 0, \quad (31.\ 2.\ \text{Cor. } 2),$$

i. e. $$S.(\beta-\alpha)\,V.V\beta\gamma V\gamma\alpha = 0.$$

Now $$V.V\beta\gamma V\gamma\alpha = -\gamma S.\beta\gamma\alpha \quad (\text{Formula } 11),$$

therefore $$S.(\beta-\alpha)\,V.V\beta\gamma V\gamma\alpha = -(S\beta\gamma - S\alpha\gamma)\,S.\beta\gamma\alpha.$$

Hence $$S\beta\gamma = S\alpha\gamma.$$

Now
$$\begin{aligned} BC^2 + OA^2 &= (\gamma-\beta)^2 + \alpha^2 \\ &= \alpha^2 + \beta^2 + \gamma^2 - 2S\beta\gamma \\ &= \alpha^2 + \beta^2 + \gamma^2 - 2S\alpha\gamma \\ &= (\gamma-\alpha)^2 + \beta^2 \\ &= AC^2 + OB^2. \end{aligned}$$

Consequently the condition that all three perpendiculars shall meet in a point is that the sum of the squares of each pair of opposite edges shall be the same.

Cor. Conversely, if the sum of the squares of each pair of opposite edges is the same, the perpendiculars from the angles on the opposite faces will meet in a point.

Ex. 3. *If P be a point in the face ABC of a tetrahedron, from which are drawn Pa, Pb, Pc, respectively parallel to OA, OB, OC to meet the opposite faces OBC, OCA, OAB in a, b, c; then will*

$$\frac{Pa}{OA} + \frac{Pb}{OB} + \frac{Pc}{OC} = 1.$$

Retaining the notation of the last examples, let $OP = \delta$, $Pa = -x\alpha$, $Pb = -y\beta$, $Pc = -z\gamma$; then

$$Oa = \delta - x\alpha,\quad Ob = \delta - y\beta,\quad Oc = \delta - z\gamma.$$

Now because P, A, B, C are in the same plane

$$S.(\delta - \alpha)(\alpha - \beta)(\beta - \gamma) = 0,$$

i. e. $$S.\delta(\alpha\beta + \beta\gamma + \gamma\alpha) = S.\alpha\beta\gamma \quad\ldots\ldots\ldots\ldots\ldots(1);$$

and because O, a, B, C are in the same plane

$$S.(\delta - x\alpha)\beta\gamma = 0,$$

i. e. $$xS\alpha\beta\gamma = S.\delta\beta\gamma \quad\ldots\ldots\ldots\ldots\ldots(2);$$

also because O, A, b, C are in the same plane

$$S.(\delta - y\beta)\gamma\alpha = 0,$$

i. e. $$yS.\beta\gamma\alpha = S.\delta\gamma\alpha,$$

or, by formula 3, $$yS.\alpha\beta\gamma = S.\delta\gamma\alpha \quad\ldots\ldots\ldots\ldots\ldots(3);$$

lastly, because O, A, B, c are in the same plane

$$S.(\delta - z\gamma)\alpha\beta = 0,$$

i. e. $$zS.\gamma\alpha\beta = S.\delta\alpha\beta,$$

or $$zS.\alpha\beta\gamma = S.\delta\alpha\beta \quad\ldots\ldots\ldots\ldots\ldots(4).$$

Adding (2), (3), and (4) there results

$$(x + y + z)S.\alpha\beta\gamma = S.\delta\beta\gamma + S.\delta\gamma\alpha + S.\delta\alpha\beta$$
$$= S.\alpha\beta\gamma, \text{ by (1)},$$

therefore $$x + y + z = 1:$$

hence $$\frac{Pa}{OA} + \frac{Pb}{OB} + \frac{Pc}{OC} = 1.$$

COR. 1. If P be in the plane ABC produced below the plane OBC, Pa as a vector will have the same sign as OA has; hence in this case we shall have

$$-\frac{Pa}{OA} + \frac{Pb}{OB} + \frac{Pc}{OC} = 1.$$

COR. 2. If P be outside both the planes OBC, OCA; we shall have

$$-\frac{Pa}{OA}-\frac{Pb}{OB}+\frac{Pc}{OC}=1.$$

Ex. 4. *Any point Q is joined to the angular points A, B, C, O of a tetrahedron, and the joining lines, produced if necessary, meet the opposite faces in a, b, c, o; to prove that*

$$\frac{Qa}{Aa}+\frac{Qb}{Bb}+\frac{Qc}{Cc}+\frac{Qo}{Oo}=1;$$

regard being had to the signs of Aa, Bb, &c., as in the last example.

Let $QA=\alpha$, $QB=\beta$, $QC=\gamma$, $QO=\delta$; $Qa=a\alpha$, $Qb=b\beta$, $Qc=c\gamma$, $Qo=d\delta$: then since a, b, c, o are points in the planes BCO, ACO, ABO, ABC, we have, as in the last example,

$$aS.\alpha(\beta\gamma+\gamma\delta+\delta\beta)=S.\beta\gamma\delta,$$

&c. &c.

i.e.

$$aS.(\alpha\beta\gamma+\alpha\gamma\delta+\alpha\delta\beta)-S.\beta\gamma\delta=0 \quad\text{............(1)},$$

$$bS.(\beta\alpha\gamma+\beta\gamma\delta+\beta\delta\alpha)-S.\alpha\gamma\delta=0 \quad\text{............(2)},$$

$$cS.(\gamma\alpha\beta+\gamma\beta\delta+\gamma\delta\alpha)-S.\alpha\beta\delta=0 \quad\text{............(3)},$$

$$dS.(\delta\alpha\beta+\delta\beta\gamma+\delta\gamma\alpha)-S.\alpha\beta\gamma=0 \quad\text{............(4)}.$$

Now, if we write

$$S.\alpha\beta\gamma=x,\quad S.\alpha\gamma\delta=y,\quad S.\alpha\delta\beta=z,\quad S.\beta\gamma\delta=u;$$

and apply the formulæ 3 and 4, we get

$$ax+ay+az-u=0,$$

$$-bx-y-bz+bu=0,$$

$$cx+cy+z-cu=0,$$

$$-x-dy-dz+du=0,$$

which give

$$\frac{a}{a-1}x+\frac{d}{d-1}u=0,$$

$$\frac{a}{a-1}y+\frac{b}{b-1}u=0,$$

$$\frac{c}{c-1}y - \frac{b}{b-1}z = 0,$$

$$\frac{c}{c-1}x - \frac{d}{d-1}z = 0;$$

and, therefore, $$\frac{1}{a-1} + \frac{b}{b-1} + \frac{c}{c-1} + \frac{d}{d-1} = 0,$$

i. e. $$\frac{a}{a-1} + \frac{b}{b-1} + \frac{c}{c-1} + \frac{d}{d-1} = 1,$$

or $$\frac{Qa}{Aa} + \frac{Qb}{Bb} + \frac{Qc}{Cc} + \frac{Qo}{Oo} = 1.$$

Ex. 5. *If two tetrahedra $ABCD$, $A'B'C'D'$ are so situated that the straight lines AA', BB', CC', DD' all meet in a point, the lines of intersection of the planes of corresponding faces shall all lie in the same plane.*

Let $A'A$, $B'B$, $C'C$, $D'D$ meet in O.

$$OA = \alpha, \quad OB = \beta, \quad OC = \gamma, \quad OD = \delta,$$

$$OA' = m\alpha, \quad OB' = n\beta, \quad OC' = p\gamma, \quad OD' = q\delta.$$

The equation of the plane ABC is (34. 5)

$$S\rho\,(V\alpha\beta + V\beta\gamma + V\gamma\alpha) = S\,.\,\alpha\beta\gamma,$$

and that of $A'B'C'$ becomes, after dividing both sides by mnp,

$$S\rho\left(\frac{1}{p}\,V\alpha\beta + \frac{1}{m}\,V\beta\gamma + \frac{1}{n}\,V\gamma\alpha\right) = S\,.\,\alpha\beta\gamma.$$

The vector line of intersection of the two planes is (34. 9)

$$V\,.\,(V\alpha\beta + V\beta\gamma + V\gamma\alpha)\left(\frac{1}{p}\,V\alpha\beta + \frac{1}{m}\,V\beta\gamma + \frac{1}{n}\,V\gamma\alpha\right),$$

i. e. by formula (11), omitting the common factor $S\,.\,\alpha\beta\gamma$,

$$\left(\frac{1}{n} - \frac{1}{p}\right)\alpha + \left(\frac{1}{p} - \frac{1}{m}\right)\beta + \left(\frac{1}{m} - \frac{1}{n}\right)\gamma.$$

From this expression the vectors of the intersections of the other planes may at once be written down.

That of ABD, $A'B'D'$ is

$$\left(\frac{1}{n}-\frac{1}{q}\right)\alpha+\left(\frac{1}{q}-\frac{1}{m}\right)\beta+\left(\frac{1}{m}-\frac{1}{n}\right)\delta\,;$$

that of ACD, $A'C'D'$ is

$$\left(\frac{1}{p}-\frac{1}{q}\right)\alpha+\left(\frac{1}{q}-\frac{1}{m}\right)\gamma+\left(\frac{1}{m}-\frac{1}{p}\right)\delta\,;$$

and that of BCD, $B'C'D'$

$$\left(\frac{1}{p}-\frac{1}{q}\right)\beta+\left(\frac{1}{q}-\frac{1}{n}\right)\gamma+\left(\frac{1}{n}-\frac{1}{p}\right)\delta.$$

Now to prove that any three of these lines lie in the same plane, all that is necessary is to prove (31. 2. Cor. 2) that the scalar of the product of their vectors equals 0.

If we take the vectors of the first three, we may write them under the form

$$a\alpha+b\beta+c\gamma,\quad a'\alpha+b'\beta+c\delta,\quad a''\alpha+b'\gamma-b\delta,$$

respectively; so that the scalar of their product is

$$S.(a\alpha+b\beta+c\gamma)\,(a'\alpha+b'\beta+c\delta)\,(a''\alpha+b'\gamma-b\delta).$$

Now the coefficient of every different scalar in this product is separately equal to 0. That of $S.\alpha\beta\gamma$ for instance is, omitting the common factor b',

$$\left(\frac{1}{n}-\frac{1}{p}\right)\left(\frac{1}{q}-\frac{1}{m}\right)-\left(\frac{1}{m}-\frac{1}{n}\right)\left(\frac{1}{p}-\frac{1}{q}\right)-\left(\frac{1}{p}-\frac{1}{m}\right)\left(\frac{1}{n}-\frac{1}{q}\right),$$

in which every term vanishes.

That again of $S.\beta\gamma\delta$ is

$$-bcb'+cb'b,$$

which is 0; and so of the rest.

Hence the intersections, two and two, of the first three pairs of planes lie in the same plane; and the same may be proved in like manner of any other three: whence the truth of the proposition

Ex. 6. *CP, CD are conjugate semi-diameters of an ellipse,*

as also CP', CD'; PP', DD' are joined; to prove that the area of the triangle PCP' equals that of the triangle DCD'.

Let α, β, α', β' be the vectors CP, CD, CP', CD'; k a unit vector perpendicular to the plane of the ellipse.

Since

$$\alpha = \psi^{-1}\psi\alpha = -(aiSi\psi\alpha + bjSj\psi\alpha), \text{ \&c., \&c. } (47.\ 5),$$

therefore
$$\begin{aligned} V\alpha\alpha' &= V.(aiSi\psi\alpha + bjSj\psi\alpha)(aiSi\psi\alpha' + bjSj\psi\alpha') \\ &= abk(Si\psi\alpha Sj\psi\alpha' - Sj\psi\alpha Si\psi\alpha') \\ &= abkS.kV(\psi\alpha\psi\alpha'). \quad \text{(Formula 16.)} \end{aligned}$$

Similarly $V\beta\beta' = abkS.kV(\psi\beta\psi\beta')$.

Now $\psi\alpha$, $\psi\beta$ are unit vectors at right angles to one another; as are also $\psi\alpha'$, $\psi\beta'$; therefore the angle between $\psi\alpha$ and $\psi\alpha'$ is the same as that between $\psi\beta$ and $\psi\beta'$.

Hence
$$S.kV(\psi\alpha\psi\alpha') = S.kV(\psi\beta\psi\beta'),$$

and
$$V\alpha\alpha' = V\beta\beta',$$

i. e. area of triangle PCP' = that of triangle DCD'.

Ex. 7. *If a parallelepiped be constructed on the semi-conjugate diameters of an ellipsoid, the sum of the squares of the areas of the faces of the parallelepiped is equal to the sum of the squares of the faces of the rectangular parallelepiped constructed on the semi-axes.*

By 63. 9,
$$\begin{aligned} \alpha &= -(aiSi\psi\alpha + bjSj\psi\alpha + ckSk\psi\alpha) \\ \beta &= -(aiSi\psi\beta + bjSj\psi\beta + ckSk\psi\beta); \end{aligned}$$

therefore
$$\begin{aligned} V\alpha\beta = abk&(Si\psi\alpha Sj\psi\beta - Si\psi\beta Sj\psi\alpha) \\ + acj&(Si\psi\alpha Sk\psi\beta - Si\psi\beta Sk\psi\alpha) \\ + bci&(Sj\psi\alpha Sk\psi\beta - Sj\psi\beta Sk\psi\alpha). \end{aligned}$$

Now $Si\psi\alpha Sj\psi\beta - Si\psi\beta Sj\psi\alpha = SVijV\psi\beta\psi\alpha$, Formula (16),

$$= -Sk\psi\gamma, \text{ (Art. 17)};$$

therefore $$V\alpha\beta = -(abkSk\psi\gamma + acjSj\psi\gamma + bciSi\psi\gamma),$$
$$V\gamma\alpha = -(abkSk\psi\beta + acjSj\psi\beta + bciSi\psi\beta),$$
$$V\beta\gamma = -(abkSk\psi\alpha + acjSj\psi\alpha + bciSi\psi\alpha).$$

If now we square and add these expressions, observing that because $\psi\alpha$, $\psi\beta$, $\psi\gamma$ are unit vectors at right angles to one another,
$$(Si\psi\alpha)^2 + (Si\psi\beta)^2 + (Si\psi\gamma)^2 = 1,$$
we shall have
$$(V\alpha\beta)^2 + (V\alpha\gamma)^2 + (V\beta\gamma)^2 = -\{(ab)^2 + (ac)^2 + (bc)^2\},$$
which (21. 4) is the proposition to be proved.

Ex. 8. *To find the locus of the intersections of tangent planes at the extremities of conjugate diameters of an ellipsoid.*

Let π be the vector to the point of intersection of tangent planes at the extremities of α, β, γ: then
$$S\pi\phi\alpha = \ \ 1, \ (57),$$
gives $$S\pi\psi^2\alpha = -1,$$
or $$S\psi\pi\psi\alpha = -1,$$
$$S\psi\pi\psi\beta = -1,$$
$$S\psi\pi\psi\gamma = -1.$$

From these three equations we extricate $\psi\pi$ by means of formula (14), which gives
$$\psi\pi S\psi\alpha\psi\beta\psi\gamma = V\psi\alpha\psi\beta S\psi\pi\psi\gamma + V\psi\beta\psi\gamma S\psi\pi\psi\alpha + V\psi\gamma\psi\alpha S\psi\pi\psi\beta;$$
therefore $$\psi\pi = V\psi\alpha\psi\beta + V\psi\beta\psi\gamma + V\psi\gamma\psi\alpha$$
$$= \psi\gamma + \psi\alpha + \psi\beta,$$
$$(\psi\pi)^2 = -(1+1+1)$$
$$= -3,$$
$$\frac{x^2}{3a^2} + \frac{y^2}{3b^2} + \frac{z^2}{3c^2} = 1;$$

an ellipsoid similar to the given ellipsoid.

Ex. 9. *If O, A, B, C, D, E are any six points in space, OX any given direction, OA', OB', OC', OD', OE' the projections of OA, OB, OC, OD, OE on OX; $BCDE$, $CDEA$, $DEAB$, $EABC$, $ABCD$ the volumes of the pyramids whose vertices are B, C, D, E, A, with a positive or negative sign in accordance with the law given in the note to* 69. 5; *then*

$$OA'.BCDE + OB'.CDEA + OC'.DEAB + OD'.EABC \\ + OE'.ABCD = 0.$$

Let OA, OB, OC, OD, OE be α, β, γ, δ, ϵ respectively.

Write for $\alpha S(\gamma-\beta)(\delta-\beta)(\epsilon-\beta)$ its value

$$\alpha(S.\gamma\delta\epsilon - S.\delta\epsilon\beta + S.\epsilon\beta\gamma - S.\beta\gamma\delta),$$

and similar expressions for $\beta S(\alpha-\gamma)(\delta-\gamma)(\epsilon-\gamma)$, &c., and there will result, by addition,

$$\alpha S(\gamma-\beta)(\delta-\beta)(\epsilon-\beta) + \beta S(\alpha-\gamma)(\delta-\gamma)(\epsilon-\gamma) \\ + \gamma S(\alpha-\delta)(\beta-\delta)(\epsilon-\delta) + \delta S(\alpha-\epsilon)(\beta-\epsilon)(\gamma-\epsilon) \\ + \epsilon S(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha) = 0,$$

i. e. retaining the notation adopted in the Note referred to,

$$OA.BCDE + OB.CDEA + OC.DEAB + OD.EABC \\ + OE.ABCD = 0.$$

Now let π be a vector along OX; then the operation by $S.\pi$ on the above expression gives the result required.

In some of the examples which follow, we will endeavour to show how a problem should *not*, as well as how it should, be attacked.

Ex. 10. *Given any three planes, and the direction of the vector perpendicular to a fourth, to find its length so that they may meet in one point.*

Let $S\alpha\rho = a$, $S\beta\rho = b$, $S\gamma\rho = c$ be the three, and let δ be the vector perpendicular to the new plane. Then, if its equation be

$$S\delta\rho = d,$$

we must find the value of d that these four equations may all be satisfied by one value of ρ.

Formula (14) gives

$$\begin{aligned} \rho S.\alpha\beta\gamma &= V\alpha\beta S\gamma\rho + V\beta\gamma S\alpha\rho + V\gamma\alpha S\beta\rho \\ &= cV\alpha\beta + aV\beta\gamma + bV\gamma\alpha, \end{aligned}$$

by the equations of the first three. Operate by $S.\delta$, and use the fourth equation, and we have the required value

$$dS.\alpha\beta\gamma = aS.\beta\gamma\delta + bS.\gamma\alpha\delta + cS.\alpha\beta\delta.$$

Ex. 11. *The sum of the (vector) areas of the faces of any tetrahedron, and therefore of any polyhedron, is zero.*

Take one corner as origin, and let α, β, γ be the vectors of the other three. Then the vector areas of the three faces meeting in the origin are

$$\frac{1}{2}V\alpha\beta, \quad \frac{1}{2}V\beta\gamma, \quad \frac{1}{2}V\gamma\alpha, \text{ respectively.}$$

That of the fourth may be expressed in any of the forms

$$\frac{1}{2}V(\gamma-\alpha)(\beta-\alpha), \quad \frac{1}{2}V(\alpha-\beta)(\gamma-\beta), \quad \frac{1}{2}V(\beta-\gamma)(\alpha-\gamma).$$

But all of these have the common value

$$\frac{1}{2}V(\gamma\beta+\beta\alpha+\alpha\gamma),$$

which is obviously the sum of the three other vector-areas taken negatively. Hence the proposition, which is an elementary one in Hydrostatics.

Now any polyhedron may be cut up by planes into tetrahedra, and the faces exposed by such treatment have vector-areas equal and opposite in sign. Hence the extension.

Ex. 12. *If the pressure be uniform throughout a fluid mass, an immersed tetrahedron (and therefore any polyhedron) experiences no couple tending to make it rotate.*

This is supplementary to the last example. The pressures on *the faces* are fully expressed by the vector areas above given, and

their points of application are the centres of inertia of the areas of the faces. The coordinates of these points are

$$\frac{1}{3}(\alpha+\beta),\ \frac{1}{3}(\beta+\gamma),\ \frac{1}{3}(\gamma+\alpha),\ \frac{1}{3}(\alpha+\beta+\gamma),$$

and the sum of the couples is

$$\begin{aligned}\frac{1}{6}V.\{V\alpha\beta.(\alpha+\beta)+V\beta\gamma.(\beta+\gamma)+V\gamma\alpha.(\gamma+\alpha)\\ +V(\gamma\beta+\beta\alpha+\alpha\gamma).(\alpha+\beta+\gamma)\}\\ =-\frac{1}{6}V(V\alpha\beta.\gamma+V\beta\gamma.\alpha+V\gamma\alpha.\beta)=0,\end{aligned}$$

by applying formula (9).

Ex. 13. *What are the conditions that the three planes*

$$S\alpha\rho=a,\quad S\beta\rho=b,\quad S\gamma\rho=c,$$

shall intersect in a straight line?

There are many ways of attacking such a question, so we will give a few for practice.

(*a*)
$$\begin{aligned}\rho S.\alpha\beta\gamma &= V\alpha\beta S\gamma\rho+V\beta\gamma S\alpha\rho+V\gamma\alpha S\beta\rho\\ &=cV\alpha\beta+aV\beta\gamma+bV\gamma\alpha\end{aligned}$$

by the given equations. But this gives a single definite value of ρ unless both sides vanish, so that the conditions are

$$S.\alpha\beta\gamma=0,$$

and
$$cV\alpha\beta+aV\beta\gamma+bV\gamma\alpha=0,$$

which *includes* the preceding.

(*b*)
$$S(l\alpha-m\beta)\rho=al-bm$$

is the equation of any plane passing through the intersection of the first two given planes. Hence, if the three intersect in a straight line there must be values of l, m such that

$$l\alpha-m\beta=\gamma,$$
$$la-mb=c.$$

The first of these gives, as before,

$$S.\alpha\beta\gamma=0,$$

and it also gives

$$V\gamma\alpha = mV\alpha\beta, \quad V\beta\gamma = -lV\alpha\beta,$$

so that if we multiply the second by $V\alpha\beta$,

$$laV\alpha\beta - mbV\alpha\beta = cV\alpha\beta$$

becomes
$$-aV\beta\gamma - bV\gamma\alpha = cV\alpha\beta;$$

the second condition of (a).

(c) Again, suppose ρ to be given by the first two in the form

$$\rho = p\alpha + q\beta + xV\alpha\beta,$$

we find
$$a = p\alpha^2 + qS\alpha\beta, \text{ because } S\alpha V\alpha\beta = 0,$$
$$b = pS\alpha\beta + q\beta^2;$$

therefore

$$\rho\begin{vmatrix} \alpha^2, & S\alpha\beta \\ S\alpha\beta, & \beta^2 \end{vmatrix} = \alpha\begin{vmatrix} a, & S\alpha\beta \\ b, & \beta^2 \end{vmatrix} + \beta\begin{vmatrix} \alpha^2, & a \\ S\alpha\beta, & b \end{vmatrix} + xV\alpha\beta,$$

so that the third equation gives, operating by $S.\gamma$,

$$c\begin{vmatrix} \alpha^2, & S\alpha\beta \\ S\alpha\beta, & \beta^2 \end{vmatrix} = S\alpha\gamma\begin{vmatrix} a, & S\alpha\beta \\ b, & \beta^2 \end{vmatrix} + S\beta\gamma\begin{vmatrix} \alpha^2, & a \\ S\alpha\beta, & b \end{vmatrix} + xS.\alpha\beta\gamma.$$

Now a determinate value of x would mean intersection in one point only; so, as before,

$$S.\alpha\beta\gamma = 0,$$
$$c(\alpha^2\beta^2 - S^2\alpha\beta) = a(\beta^2S\alpha\gamma - S\alpha\beta S\beta\gamma) - b(S\alpha\beta S\alpha\gamma - \alpha^2S\beta\gamma).$$

The latter may be written

$$S.\alpha[c(\alpha\beta^2 - \beta S\alpha\beta) - a(\gamma\beta^2 - \beta S\beta\gamma) - b(\alpha S\beta\gamma - \gamma S\alpha\beta)] = 0.$$

Now
$$\begin{aligned} S.\alpha(\alpha\beta^2 - \beta S\alpha\beta) &= S\alpha(\beta.\beta\alpha - \beta S\beta\alpha) \\ &= S.\alpha(\beta V\beta\alpha) \\ &= -S.\alpha(\beta V\alpha\beta) = -S(\alpha\beta V\alpha\beta). \end{aligned}$$

Similarly,
$$S.\alpha(\gamma\beta^2 - \beta S\beta\gamma) = S(\alpha\beta V\beta\gamma),$$

and
$$\begin{aligned} S.\alpha(\alpha S\beta\gamma - \gamma S\alpha\beta) &= S.\alpha(V.\beta V\gamma\alpha), \text{ (formula 8)}, \\ &= S(\alpha\beta V\gamma\alpha). \end{aligned}$$

The equation now becomes

$$S.\alpha\beta(cV\alpha\beta + aV\beta\gamma + bV\gamma\alpha) = 0.$$

Now since $S.\alpha\beta\gamma = 0$, α, β, γ are vectors in the same plane; therefore γ may be written $m\alpha + n\beta$,

$$\text{and } cV\alpha\beta + aV\beta\gamma + V\gamma\alpha$$

assumes the form $eV\alpha\beta$, which, unless $e = 0$, gives

$$S(\alpha\beta V\alpha\beta) = 0,$$

or $V\alpha\beta$ is in the same plane with α, β; but it is also perpendicular to the plane, which is absurd; therefore $e = 0$, or

$$cV\alpha\beta + aV\beta\gamma + bV\gamma\alpha = 0\,;$$

thus the third and prolix method leads to the same conclusion as the first.

Ex. 14. *Find the surface traced out by a straight line which remains always perpendicular to a given line while intersecting each of two fixed lines.*

Let the equations of the fixed lines be

$$\varpi = \alpha + x\beta, \quad \varpi_1 = \alpha_1 + x_1\beta_1.$$

Then if ρ be the vector of the new line in any position

$$\begin{aligned}\rho &= \varpi + y(\varpi_1 - \varpi)\\ &= (1-y)(\alpha + x\beta) + y(\alpha_1 + x_1\beta_1).\end{aligned}$$

This is not, as yet, the equation required. For it involves essentially *three* independent constants, x, x_1, y; and may therefore in general be made to represent any point whatever of infinite space. The reader may easily see this if he reflects that two lines which are not parallel must appear, from every point of space, to intersect one another. We have still to introduce the condition that the new line is perpendicular to a fixed vector, γ suppose, which gives

$$S.\gamma(\varpi_1 - \varpi) = 0 = S.\gamma[(\alpha_1 - \alpha) + x_1\beta_1 - x\beta].$$

This gives x_1 in terms of x, so that there are now but two indeterminates in the equation for ρ, which therefore represents a surface, which, it is not difficult to see, is one of the second order.

Ex. 15. *Find the condition that the equation*

$$S.\rho\phi\rho = 1$$

may represent a surface of revolution.

The expression $\phi\rho$ here stands for something more general than that employed in Chap. VIII. above, in fact it may be written

$$\phi\rho = \alpha S\alpha_1\rho + \beta S\beta_1\rho + \gamma S\gamma_1\rho,$$

where α, α_1, β, β_1, γ, γ_1 are any six vectors whatever. This will be more carefully examined in the next chapter.

If the surface be one of revolution then, since it is central and of the second degree, it is obvious that any sphere whose centre is at the origin will cut it in two equal circles in planes perpendicular to the axis, and that these will be equidistant from the origin. Hence, if r be the radius of one of these circles, ϵ the vector to its centre, ρ the vector to any point in its circumference, it is evident that we have the following equation

$$S\rho\phi\rho - 1 - C(\rho^2 + r^2) = (S\epsilon\rho)^2 - e^2,$$

where C and e are constants. This, being an identity, gives

$$\left.\begin{array}{l} 1 - e^2 + Cr^2 = 0 \\ S\rho\phi\rho - C\rho^2 = (S\epsilon\rho)^2 \end{array}\right\}.$$

The form of these equations shows that C is an absolute constant, while r and e are related to one another by the first; and the second gives

$$\phi\rho = C\rho + \epsilon S\epsilon\rho.$$

This shows simply that $\quad S.\epsilon\rho\phi\rho = 0,$

i. e. ϵ, ρ, and $\phi\rho$ are coplanar, i. e. all the normals pass through a given straight line; or that the expression

$$V\rho\phi\rho,$$

whatever be ρ, expresses always a vector parallel to a particular plane.

Ex. 16. *If three mutually perpendicular vectors be drawn from a point to a plane, the sum of the reciprocals of the squares of their lengths is independent of their directions.*

Let $$S\epsilon\rho = 1$$
be the equation of the plane, and let α, β, γ be any set of mutually perpendicular unit-vectors. Then, if $x\alpha$, $y\beta$, $z\gamma$ be points in the plane, we have
$$xS\alpha\epsilon = 1, \quad yS\beta\epsilon = 1, \quad zS\gamma\epsilon = 1,$$
whence $$-\epsilon = \alpha S\alpha\epsilon + \beta S\beta\epsilon + \gamma S\gamma\epsilon \ (63.\ 2) = \frac{\alpha}{x} + \frac{\beta}{y} + \frac{\gamma}{z}.$$
Taking the tensor, we have
$$T\epsilon^2 = \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}.$$

Ex. 17. *Find the equation of the straight line which meets, at right angles, two given straight lines.*

Let $$\varpi = \alpha + x\beta, \quad \varpi = \alpha_1 + x_1\beta_1,$$
be the two lines; then the equation of the required line must be of the form
$$\varpi = \alpha_2 + x_2 V\beta\beta_1,$$
and nothing is undetermined but α_2.

Since the first and third equations denote lines having one point in common, we have
$$S.\beta V\beta\beta_1 (\alpha - \alpha_2) = 0.$$
Similarly $$S.\beta_1 V\beta\beta_1 (\alpha_1 - \alpha_2) = 0.$$
Let $$\alpha_2 = y\beta + y_1\beta_1,$$
(it is obviously superfluous to add a term in $V\beta\beta_1$), then
$$S.\alpha\beta V\beta\beta_1 = y_1 T^2 V\beta\beta_1,$$
$$S.\alpha_1\beta_1 V\beta\beta_1 = -yT^2 V\beta\beta_1,$$
and, finally,
$$\varpi = \frac{1}{T^2 V\beta\beta_1}(\beta_1 S.\alpha\beta V\beta\beta_1 - \beta S.\alpha_1\beta_1 V\beta\beta_1) + x_2 V\beta\beta_1.$$

Ex. 18. *If* $T\rho = T\alpha = T\beta = 1$, *and* $S.\alpha\beta\rho = 0$, *show that*
$$S.U(\rho - \alpha)\,U(\rho - \beta) = \sqrt{\frac{1}{2}(1 - S\alpha\beta)}.$$

Interpret this theorem geometrically.

We have, from the given equations, the following, which are equivalent to them,

$$\left.\begin{aligned} \rho^2 = \alpha^2 = \beta^2 = -1 \\ \rho = x\alpha + y\beta \end{aligned}\right\}.$$

Hence
$$-x^2 - y^2 + 2xyS\alpha\beta = -1,$$

$$U(\rho - \alpha) = \frac{(x-1)\alpha + y\beta}{\sqrt{(x-1)^2 - 2(xy - y)S\alpha\beta + y^2}},$$

$$U(\rho - \beta) = \frac{x\alpha + (y-1)\beta}{\sqrt{x^2 - 2(xy - x)S\alpha\beta + (y-1)^2}},$$

$$S.U(\rho - \alpha)\,U(\rho - \beta)$$

$$= \frac{-x(x-1) + [xy + (x-1)(y-1)]S\alpha\beta - y(y-1)}{\sqrt{x^2 + y^2 - 2x + 1 - 2(xy - y)S\alpha\beta}\;\sqrt{x^2 + y^2 - 2y + 1 - 2(xy - x)S\alpha\beta}}$$

$$= \frac{x + y - (x + y - 1)S\alpha\beta - 1}{\sqrt{2 - 2x + 2yS\alpha\beta}\;\sqrt{2 - 2y + 2xS\alpha\beta}}$$

$$= \frac{(x + y - 1)(1 - S\alpha\beta)}{2\sqrt{(1 - x - y)(1 - S\alpha\beta) + xy\{1 - (S\alpha\beta)^2\}}}$$

$$= \frac{x + y - 1}{2}\sqrt{\frac{1 - S\alpha\beta}{1 - x - y + xy(1 + S\alpha\beta)}}$$

$$= \frac{x + y - 1}{2}\sqrt{\frac{1 - S\alpha\beta}{1 - x - y + \frac{1}{2}(2xy + x^2 + y^2 - 1)}}$$

$$= \frac{x + y - 1}{\sqrt{2}}\sqrt{\frac{1 - S\alpha\beta}{1 - 2(x + y) + x^2 + y^2 + 2xy}}$$

$$= \pm\sqrt{\frac{1}{2}(1 - S\alpha\beta)}.$$

Of course there are far simpler solutions. Thus, for instance, the given equations show that ρ, α, β are radii of some unit *circle*. Hence the expression is the cosine of the supplement of the angle between two chords of a circle drawn from the same point in the circumference. This is obviously half the angle

subtended at the centre by radii drawn to the other ends of the chords. The cosine of this angle is

$$-S\alpha\beta,$$

and therefore the cosine of its half is

$$\sqrt{\frac{1}{2}(1-S\alpha\beta)}.$$

Ex. 19. *Find the relative position, at any instant, of two points, which are moving uniformly in straight lines.*

If α', β' be their vector velocities, t the time elapsed since their vectors were α, β, their relative vector is

$$\begin{aligned}\rho &= \alpha + t\alpha' - \beta - t\beta' \\ &= (\alpha - \beta) + t(\alpha' - \beta'),\end{aligned}$$

so that relatively to one another the motion is rectilinear, and the vector velocity is

$$\alpha' - \beta'.$$

To find the time at which the mutual distance is least.

Here we may write

$$\begin{aligned}\rho &= \gamma + t\delta, \\ T\rho^2 &= -\gamma^2 - 2tS\gamma\delta - t^2\delta^2 \\ &= \frac{(S\gamma\delta)^2}{\delta^2} - \gamma^2 - \delta^2\left(t + \frac{S\gamma\delta}{\delta^2}\right)^2.\end{aligned}$$

As the last term is positive, this is least when it vanishes, i.e. when

$$t = -S.\gamma\delta^{-1}.$$

This gives

$$\begin{aligned}\rho &= \gamma - \delta S\gamma\delta^{-1} \\ &= \gamma V\delta^{-1}\gamma,\end{aligned}$$

the vector perpendicular drawn to the relative path; as is, of course, self-evident.

Ex. 20. *Find the locus of a given point in a line of given length, when the extremities of the line move in circles in one plane.* (Watt's *Parallel Motion.*)

Let σ and τ be the vectors of the ends of the line, drawn from the centres α, β of the circles. Then if ρ be the vector of the required point

$$\rho = (\alpha + \sigma)(1 - e) + e(\beta + \tau),$$

subject to the conditions

$$\{\alpha + \sigma - (\beta + \tau)\}^2 = -l^2,$$

$$S\gamma\sigma = 0, \quad S\gamma\tau = 0,$$

$$\sigma^2 = -a^2, \quad \tau^2 = -b^2.$$

From these equations σ and τ must be eliminated. We leave the work to the reader. There is obviously an equation of condition

$$S.\gamma(\beta - \alpha) = 0.$$

Ex. 21. *Classify the curves represented by an equation of the form*

$$\rho = \frac{\alpha + x\beta + x^2\gamma}{a + bx + cx^2},$$

where α, β, γ are given vectors, and a, b, c given scalars.

In the first place we remark that x^2 in the numerator merely adds a constant vector to the value of ρ, unless $c = 0$.

Thus, if c do not vanish, the equation may be written with a change of α and β, and in general a change of origin,

$$\rho = \frac{\alpha + x\beta}{a + bx + cx^2}:$$

and this again, by change of x and of α and β, as

$$\rho = \frac{\alpha + x\beta}{a + cx^2}.$$

It is obvious that this represents a plane curve.

Also

$$\frac{S\alpha\rho}{S\beta\rho} = \frac{\alpha^2 + xS\alpha\beta}{S\alpha\beta + x\beta^2}:$$

Hence both numerator and denominator of x are of the first degree in $S\alpha\rho$, $S\beta\rho$; and therefore

$$S\alpha\rho = \frac{\alpha^2 + xS\alpha\beta}{a + cx^2}$$

gives an equation of the third degree in ρ by the elimination of x.

When we have $$S\alpha\beta = 0,$$

$$S\alpha\rho = \frac{\alpha^2}{a + cx^2},$$

$$S\beta\rho = \frac{x\beta^2}{a + cx^2},$$

whence $$x = \frac{\alpha^2 S\beta\rho}{\beta^2 S\alpha\rho},$$

and $$a\,(S\alpha\rho)^2 + c\,\frac{\alpha^4}{\beta^4}\,(S\beta\rho)^2 = \alpha^2 S\alpha\rho,$$

a conic section.

If $c = 0$, then with a change of x, α, β, γ, the equation may be written

$$\rho = \frac{\alpha}{x} + \beta + x\gamma,$$

a hyperbola—so long at least as b does not also vanish.

If b and c both vanish, the equation is obviously that of a parabola.

If a and b both vanish, whilst c has a real value, we have again a parabola.

If a vanish while b and c have real values, we have again a hyperbola.

Ex. 22. *Find the locus of a point at which a given finite straight line subtends a given angle.*

Take the middle point of the line as origin, and let $\pm\alpha$ be the vectors of its ends. At ρ it subtends an angle whose cosine is

$$-SU(\rho-\alpha)\,U(\rho+\alpha).$$

This, equated to a constant, gives the locus required. We may write the equation

$$\alpha^2-\rho^2=cT(\rho-\alpha)\,T(\rho+\alpha).$$

This is, obviously, a surface of the fourth order; a ring or tore formed by the rotation of a circle about a chord. When $c=0$, i.e. when the angle is a right angle, the two sheets of this surface close up into the sphere

$$\rho^2=\alpha^2.$$

A plane section (in the plane α, β (suppose) where $T\beta=T\alpha$ and $S\alpha\beta=0$) gives

$$\rho=x\alpha+y\beta,$$

$$\{\alpha^2(1-x^2)-y^2\alpha^2\}^2=c^2\{(x-1)^2+y^2\}\{(x+1)^2+y^2\}\,\alpha^4,$$

or
$$\{1-(x^2+y^2)\}^2=c^2\{(x^2+y^2+1)^2-4x^2\},$$

or, finally,
$$1-(x^2+y^2)=\pm\frac{2cy}{\sqrt{1-c^2}},$$

which, of course, denotes two equal circles intersecting at the ends of the fixed line.

Additional Examples to Chap. IX.

1. Prove that $S.(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)=2S.\alpha\beta\gamma$.

2. $S.V\alpha\beta V\beta\gamma V\gamma\alpha=-(S\alpha\beta\gamma)^2$.

3. $S.V(V\alpha\beta V\beta\gamma)\,V(V\beta\gamma V\gamma\alpha)\,V(V\gamma\alpha V\alpha\beta)=-(S.\alpha\beta\gamma)^4$.

4. $S(V\beta\gamma V\gamma\alpha)=\gamma^2 S\alpha\beta-S\beta\gamma S\gamma\alpha$.

5. $\alpha^2\beta^2\gamma^2=(V\alpha\beta\gamma)^2-(S\alpha\beta\gamma)^2$

6. $\quad=\alpha^2(S\beta\gamma)^2+\beta^2(S\gamma\alpha)^2+\gamma^2(S\alpha\beta)^2-(S\alpha\beta\gamma)^2$
$\quad-2S\alpha\beta S\beta\gamma S\gamma\alpha$.

7. $S(\gamma V . \alpha\beta\gamma) = \gamma^2 S\alpha\beta$.

8. $(\alpha\beta\gamma)^2 = \alpha^2\beta^2\gamma^2 + 2\alpha\beta\gamma S . \alpha\beta\gamma$.

9. $S(V\alpha\beta\gamma V\beta\gamma\alpha V\gamma\alpha\beta) = 4S\alpha\beta S\beta\gamma S\gamma\alpha S . \alpha\beta\gamma$.

10. The expression

$$V\alpha\beta V\gamma\delta + V\alpha\gamma V\delta\beta + V\alpha\delta V\beta\gamma$$

denotes a vector. What vector?

(Tait's *Quaternions*. Miscellaneous Ex. 1.)

11. $S\alpha\rho S . \beta\gamma\delta - S\beta\rho S . \gamma\delta\alpha + S\gamma\rho S . \delta\alpha\beta - S\delta\rho S . \alpha\beta\gamma = 0$.

12. $(\alpha\beta\gamma)^2 = 2\alpha^2\beta^2\gamma^2 + \alpha^2(\beta\gamma)^2 + \beta^2(\alpha\gamma)^2 + \gamma^2(\alpha\beta)^2 - 4\alpha\gamma S\alpha\beta S\beta\gamma$.

(Hamilton, *Elements*, p. 346.)

13. With the notation of the Note, Art. 69. 5, we shall have

$$DABC = OABC - OBCD + OCDA - ODAB.$$

14. When A, B, C, D are in the same plane,

$$\alpha . BCD - \beta . CDA + \gamma . DAB - \delta . ABC = 0,$$

where BCD, &c. are the areas of the triangles.

15. $\delta V . \alpha\beta\gamma + \alpha V . \beta\gamma\delta + \beta V . \gamma\delta\alpha + \gamma V . \delta\alpha\beta = 4S . \alpha\beta\gamma\delta$.

CHAPTER X.

VECTOR EQUATIONS OF THE FIRST DEGREE.

WITH the object of giving the student an idea of one of the physical applications of Quaternions, we will treat the solution of linear and vector equations from an elementary kinematical point of view.

DEF. *Homogeneous Strain* is such that portions of a body, originally equal, similar, and similarly placed, remain after the strain equal, similar, and similarly placed.

Thus straight lines remain straight lines, parallel lines remain parallel, equal parallel lines remain equal, planes remain planes, parallel planes remain parallel, and equal areas on parallel planes remain equal. Also the volumes of *all* portions of the body are increased or diminished in the same proportion, as is easily seen by supposing the body originally divided into small equal cubes by series of planes perpendicular to each other.

It is thus obvious that a homogeneous strain is entirely determined if we know into what vectors three given (non-coplanar) vectors are changed by it. Thus if α, β, γ become α', β', γ' respectively, any other vector which may be expressed as

$$\rho = \frac{1}{S \,.\, \alpha\beta\gamma}(\alpha S \,.\, \beta\gamma\rho + \beta S \,.\, \gamma\alpha\rho + \gamma S \,.\, \alpha\beta\rho)$$

is changed to

$$\rho' = \frac{1}{S \,.\, \alpha\beta\gamma}(\alpha' S \,.\, \beta\gamma\rho + \beta' S \,.\, \gamma\alpha\rho + \gamma' S \,.\, \alpha\beta\rho).$$

No needful generality is lost, while much simplification is gained, by taking α, β, γ as unit vectors at right angles to one another. We thus have

$$\rho = -(\alpha S\alpha\rho + \beta S\beta\rho + \gamma S\gamma\rho),$$
$$\rho' = -(\alpha' S\alpha\rho + \beta' S\beta\rho + \gamma' S\gamma\rho).$$

Comparing these expressions we see that *Homogeneous Strain* alters a vector into a definite linear and vector function of its original value.

In abbreviated notation, we may write (as in Art. 63, though our symbol, as will soon be seen, is more general than that there employed)

$$\phi\rho = -(\alpha' S\alpha\rho + \beta' S\beta\rho + \gamma' S\gamma\rho),$$

where ϕ itself depends upon *nine* independent constants involved in the three equations

$$\left.\begin{aligned} \phi\alpha &= \alpha' \\ \phi\beta &= \beta' \\ \phi\gamma &= \gamma' \end{aligned}\right\}.$$

For α', β', γ' may of course be expressed in terms of α, β, γ: and, as they are quite independent of one another, the nine coefficients in the following equations may have absolutely any values whatever;

$$\left.\begin{aligned} \phi\alpha &= \alpha' = A\alpha + c\beta + b'\gamma \\ \phi\beta &= \beta' = c'\alpha + B\beta + a\gamma \\ \phi\gamma &= \gamma' = b\alpha + a'\beta + C\gamma \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (a).$$

In discussing the particular form of ϕ which occurs in the treatment of central surfaces of the second order we found, Art. 44, that it possessed the property

$$S.\sigma\phi\rho = S.\rho\phi\sigma \ldots\ldots\ldots\ldots\ldots\ldots\ldots (b),$$

whatever vectors are represented by ρ and σ. Remembering that α, β, γ form a rectangular unit system, we find from (a)

$$\left.\begin{aligned} S.\beta\phi\alpha &= -c \\ S.\alpha\phi\beta &= -c' \end{aligned}\right\},$$

with other similar pairs; so that our new value of ϕ satisfies (*b*) if, and only if, we have in (*a*)

$$\left.\begin{array}{l} a=a' \\ b=b' \\ c=c' \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(c).$$

The physical meaning of this condition will be seen immediately.

But, although (*b*) is not generally true, we have

$$\begin{aligned} S\,.\,\sigma\phi\rho &= -\left(S\alpha'\sigma S\alpha\rho + S\beta'\sigma S\beta\rho + S\gamma'\sigma S\gamma\rho\right) \\ &= -S\,.\,\rho\left(\alpha S\alpha'\sigma + \beta S\beta'\sigma + \gamma S\gamma'\sigma\right), \end{aligned}$$

where the expression in brackets is a linear and vector function of σ, depending upon the same *nine* scalars as those in ϕ; and which we may therefore express by ϕ', so that

$$\phi'\sigma = -\left(\alpha S\alpha'\sigma + \beta S\beta'\sigma + \gamma S\gamma'\sigma\right) \ldots\ldots\ldots\ldots(d).$$

And with this we have obviously

$$S\,.\,\sigma\phi\rho = S\,.\,\rho\phi'\sigma \ldots\ldots\ldots\ldots\ldots\ldots\ldots(e),$$

which is the general relation, of which (*b*) is a mere particular case.

By putting α, β, γ in succession for σ in (*d*) and referring to (*a*) we have

$$\left.\begin{array}{l} \phi'\alpha = A\alpha + c'\beta + b\gamma \\ \phi'\beta = c\alpha + B\beta + a'\gamma \\ \phi'\gamma = b'\alpha + a\beta + C\gamma \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(f).$$

Comparing (*f*) with (*a*) we see that

$$\phi\rho = \phi'\rho,$$

whatever be ρ, provided the conditions (*c*) be fulfilled. This agrees with the result already obtained.

Either of the functions ϕ and ϕ', thus defined together, is called the *Conjugate* of the other: and when they are equal (i.e. when (*c*) is satisfied) ϕ is called a *Self-Conjugate* function. As we

employed it in Chap. VI, ϕ was self-conjugate; and, even had it not been so, it was involved (as we shall presently see) in such a manner that its non-conjugate part was necessarily absent.

We may now write, as before,

$$\phi\rho = -(\alpha' S\alpha\rho + \beta' S\beta\rho + \gamma' S\gamma\rho),$$

and, by (d),

$$\phi'\rho = -(\alpha S\alpha'\rho + \beta S\beta'\rho + \gamma S\gamma'\rho).$$

From these we have by subtraction,

$$\begin{aligned}(\phi - \phi')\rho = \phi\rho - \phi'\rho &= \alpha S\alpha'\rho - \alpha' S\alpha\rho + \beta S\beta'\rho - \beta' S\beta\rho + \gamma S\gamma'\rho - \gamma' S\gamma\rho \\ &= V.V\alpha\alpha'\rho + V.V\beta\beta'\rho + V.V\gamma\gamma'\rho \\ &= 2V.\epsilon\rho \quad\quad (g);\end{aligned}$$

if we agree to write

$$2\epsilon = V(\alpha\alpha' + \beta\beta' + \gamma\gamma') \quad\quad (h).$$

We may now express that ϕ is self-conjugate by writing

$$\epsilon = 0,$$

the physical interpretation of which equation is of the highest importance, as will soon appear.

If we form by means of (a) the value of ϵ as in (h) we get

$$\begin{aligned}2\epsilon &= (c\gamma - b'\beta) + (a\alpha - c'\gamma) + (b\beta - a'\alpha) \\ &= (a - a')\alpha + (b - b')\beta + (c - c')\gamma,\end{aligned}$$

which obviously cannot vanish unless (as before) the three conditions (c) are satisfied.

By adding the values of $\phi\rho$ and $\phi'\rho$ above we obtain

$$\begin{aligned}(\phi + \phi')\rho = \phi\rho + \phi'\rho &= -(\alpha S\alpha'\rho + \alpha' S\alpha\rho + \beta S\beta'\rho + \beta' S\beta\rho + \gamma S\gamma'\rho + \gamma' S\gamma\rho) \\ &= -V(\alpha\rho\alpha' + \beta\rho\beta' + \gamma\rho\gamma') - \rho(S\alpha\alpha' + S\beta\beta' + S\gamma\gamma').\end{aligned}$$

As we have (by 69. 6)

$$V.\alpha\rho\alpha' = V.\alpha'\rho\alpha, \ \&c.,$$

this new function of ρ is self-conjugate; as will easily be seen by putting it for ϕ in (b) and remembering that (by 69. 17) we have

$$S.\sigma\alpha\rho\alpha' = S.\rho\alpha'\sigma\alpha = S.\rho\alpha\sigma\alpha', \ \&c., \&c.$$

Hence we may write

$$(\phi + \phi')\,\rho = 2\overline{\varpi}\rho \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i),$$

where the bar over $\overline{\varpi}$ signifies that it is self-conjugate, and the factor 2 is introduced for convenience.

From (g) and (i) we have

$$\left.\begin{aligned}\phi\rho &= \overline{\varpi}\rho + V\epsilon\rho \\ \phi'\rho &= \overline{\varpi}\rho - V\epsilon\rho\end{aligned}\right\} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (j).$$

If instead of $\phi\rho$ in any of the above investigations we write $(\phi + g)\,\rho$, it is obvious that $\phi'\rho$ becomes $(\phi' + g)\,\rho$: and the only change in the coefficients in (a) and (f) is the addition of g to each of the main series A, B, C.

We now come to Hamilton's grand proposition with regard to linear and vector functions. If ϕ be such that, in general, the vectors

$$\rho, \;\phi\rho, \;\phi^2\rho$$

(where $\phi^2\rho$ is an abbreviation for $\phi\,(\phi\rho)$) are not in one plane, then any fourth vector such as $\phi^3\rho$ (a contraction for $\phi\,(\phi\,(\phi\rho))$) can be expressed in terms of them as in 31. 5.

Thus $$\phi^3\rho = m_2\phi^2\rho - m_1\phi\rho + m\rho \;\ldots\ldots\ldots\ldots\ldots\ldots (k),$$

where m, m_1, m_2 are scalars whose values will be found immediately. That they are independent of ρ is obvious, for we may put α, β, γ in succession for ρ and thus obtain three equations of the form

$$\phi^3\alpha = m_2\phi^2\alpha - m_1\phi\alpha + m\alpha \;\ldots\ldots\ldots\ldots\ldots\ldots (l),$$

from which their values can be found. Conversely, if quantities m, m_1, m_2 can be found which satisfy (l), we may reproduce (k) by putting

$$\rho = x\alpha + y\beta + z\gamma$$

and adding together the three expressions (l) multiplied by x, y, z respectively. For it is obvious from the expression for ϕ that

$$x\phi\rho = \phi\,(x\rho), \quad x\phi^2\rho = \phi^2\,(x\rho), \;\; \&c.,$$

whatever scalar be represented by x.

If ρ, $\phi\rho$, and $\phi^2\rho$ are in the same plane, then applying the strain ϕ again we find $\phi\rho$, $\phi^2\rho$, $\phi^3\rho$ in one plane; and thus equation (k) holds for this case also. And it of course holds if $\phi\rho$ is parallel to ρ, for then $\phi^2\rho$ and $\phi^3\rho$ are also parallel to ρ.

We will prove that scalars can be found which satisfy the three equations (l) (equivalent to *nine* scalar equations, of which, however, six depend upon the other three) by actually determining their values.

The volume of the parallelepiped whose three conterminous edges are λ, μ, ν is (31. 1)

$$S.\lambda\mu\nu.$$

After the strain its volume is

$$S.\phi\lambda\phi\mu\phi\nu,$$

so that the ratio

$$\frac{S.\phi\lambda\phi\mu\phi\nu}{S.\lambda\mu\nu}$$

is the same whatever vectors λ, μ, ν may be; and depends therefore on the constants of ϕ alone. We may therefore assume

$$\left.\begin{aligned}\lambda &= \rho,\\ \mu &= \phi\rho,\\ \nu &= \phi^2\rho,\end{aligned}\right\}$$

and by inspection of (k) we find

$$\frac{S.\phi\lambda\phi\mu\phi\nu}{S.\lambda\mu\nu} = \frac{S.\phi\rho\phi^2\rho\phi^3\rho}{S.\rho\phi\rho\phi^2\rho} = m \text{...............} (m),$$

which gives the physical meaning of this constant in (k). As we may put if we please

$$\left.\begin{aligned}\lambda &= \alpha,\\ \mu &= \beta,\\ \nu &= \gamma,\end{aligned}\right\}$$

we see by (a) that

$$m = \frac{S.\phi\alpha\phi\beta\phi\gamma}{S.\alpha\beta\gamma} = \begin{vmatrix} A, & c, & b' \\ c', & B, & a \\ b, & a', & C \end{vmatrix},$$

which is the expression for the ratio in which the volume of each portion has been increased. This is unchanged by putting ϕ' for ϕ; for it becomes, by (f),

$$m = \begin{vmatrix} A, & c', & b \\ c, & B, & a' \\ b', & a, & C \end{vmatrix}.$$

Recurring to (m) we may write it by (e) as

$$S.\lambda\phi' V\phi\mu\phi\nu = mS.\lambda V\mu\nu,$$

from which, as λ is *absolutely any vector*, we have

$$\left.\begin{array}{l}\phi' V\phi\mu\phi\nu = mV\mu\nu \\ \text{or} \quad \phi\, V\phi'\mu\phi'\nu = mV\mu\nu\end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(n).$$

In passing we may notice that (n) gives us the complete solution of a linear and vector equation such as

$$\phi\sigma = \delta,$$

where δ and ϕ are given and σ is to be found. We have in fact only to take any two vectors μ and ν which are perpendicular to δ, and such that

$$V\mu\nu = \delta,$$

and we have for the unknown vector

$$\sigma = \frac{1}{m} V\phi'\mu\phi'\nu,$$

which can be calculated, as ϕ is given.

If in (n) we put $\phi + g$ for ϕ we must do so for the value of m in (m). Calling the latter M_g we have

$$\begin{aligned} M_g &= \frac{S.(\phi+g)\lambda(\phi+g)\mu(\phi+g)\nu}{S.\lambda\mu\nu} \\ &= m + g\frac{S.\lambda\phi\mu\phi\nu + S.\mu\phi\nu\phi\lambda + S.\nu\phi\lambda\phi\mu}{S.\lambda\mu\nu} \\ &\quad + g^2\frac{S.\lambda\mu\phi\nu + S.\nu\lambda\phi\mu + S.\mu\nu\phi\lambda}{S.\lambda\mu\nu} \\ &\quad + g^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(o), \end{aligned}$$

and by (n) $\quad (\phi+g)\,V(\phi'+g)\mu(\phi'+g)\nu = M_g.V\mu\nu \ldots\ldots\ldots(p),$

or
$$\left.\begin{aligned} M_g &= m + \mu_1 g + \mu_2 g^2 + g^3 \\ (\phi+g)[m\phi^{-1}V\mu\nu + g\,(V\phi'\mu\,\nu + V\mu\phi'\nu) + g^2 V\mu\nu] &= M_g V\mu\nu \end{aligned}\right\}\ldots(q).$$

From the latter of these equations it is obvious that
$$V\phi'\mu\,\nu + V\mu\phi'\nu$$
must be a linear and vector function of $V\mu\nu$, since all the other terms of the equation are such functions.

As practice in the use of these functions we will solve a problem of a little greater generality. The vectors
$$V\mu\nu, \quad V\phi'\mu\,\nu, \text{ and } V\mu\phi'\nu$$
are not generally coplanar. In terms of these (31. 5), let us express $\phi V\mu\nu$.

Let $$\phi V\mu\nu = xV\mu\nu + yV\phi'\mu\nu + zV\mu\phi'\nu.$$
Operate by $S.\lambda$, $S.\mu$, $S.\nu$ successively, then
$$S.\mu\nu\phi'\lambda = xS.\lambda\mu\nu + yS.\nu\lambda\phi'\mu + zS.\lambda\mu\phi'\nu,$$
$$S.\mu\nu\phi'\mu = yS.\nu\mu\phi'\mu,$$
$$S.\mu\nu\phi'\nu = zS.\nu\mu\phi'\nu.$$
The two last equations give (by 69. 4)
$$y = -1, \quad z = -1,$$
and therefore the first gives
$$x = \frac{S.\mu\nu\phi'\lambda + S.\nu\lambda\phi'\mu + S.\lambda\mu\phi'\nu}{S.\lambda\mu\nu}$$
$$= \mu_2, \text{ by } (q).$$

Hence, finally,
$$\phi V\mu\nu = \mu_2 V\mu\nu - V\phi'\mu\,\nu - V\mu\phi'\nu \ldots\ldots\ldots\ldots\ldots(r).$$

Substituting this in (q), and putting σ for $V\mu\nu$, which is any vector whatever, we have
$$(\phi+g)\,[m\phi^{-1} + g\,(\mu_2 - \phi) + g^2]\,\sigma = (m + \mu_1 g + \mu_2 g^2 + g^3)\,\sigma,$$
or, multiplying out,
$$(m - g\phi^2 + \mu_2 g\phi - g^2\phi + gm\phi^{-1} + g^2\phi + g^2\mu_2 + g^3)\,\sigma$$
$$= (m + \mu_1 g + \mu_2 g^2 + g^3)\,\sigma\,;$$

that is $$(-\phi^2+\mu_2\phi+m\phi^{-1})\sigma=\mu_1\sigma,$$

or $$(\phi^3-\mu_2\phi^2+\mu_1\phi-m)\sigma=0.$$

Comparing this with (k) we see that

$$\left.\begin{aligned} m_2=\mu_2&=\frac{S.\lambda\mu\phi\nu+S.\nu\lambda\phi\mu+S.\mu\nu\phi\lambda}{S.\lambda\mu\nu} \\ m_1=\mu_1&=\frac{S.\lambda\phi\mu\phi\nu+S.\mu\phi\nu\phi\lambda+S.\nu\phi\lambda\phi\mu}{S.\lambda\mu\nu} \end{aligned}\right\}\quad\ldots\ldots\ldots(s),$$

and thus the determination is complete.

We may write (k), if we please, in the form

$$m\phi^{-1}\rho=m_1\rho-m_2\phi\rho+\phi^2\rho,\quad\ldots\ldots\ldots\ldots\ldots(k'),$$

which gives another, and more direct, solution of the equation (above mentioned)

$$\phi\sigma=\delta.$$

Physically, the result we have arrived at is the solution of the problem, "By adding together scalar multiples of any vector of a body, of the corresponding vector of the same strained homogeneously, and of that of the same twice over strained, to represent the state of the body which would be produced by supposing the strain to be reversed or inverted."

These properties of the function ϕ are sufficient for many applications, and we proceed to give a few.

I. Homogeneous strain converts an originally spherical portion of a body into an ellipsoid.

For if ρ be a radius of the sphere, σ the vector into which it is changed by the strain, we have

$$\sigma=\phi\rho,$$

and $$T\rho=C,$$

from which we obtain

$$T\phi^{-1}\sigma=C,$$

or $$S.\phi^{-1}\sigma\phi^{-1}\sigma=-C^2,$$

or, finally, $$S.\sigma\phi^{-2}\sigma=-C^2.$$

This is the equation of a central surface of the second degree; and, therefore, of course, from the nature of the problem, an ellipsoid.

II. To find the vectors whose direction is unchanged by the strain.

Here $\phi\rho$ must be parallel to ρ or

$$\phi\rho = g\rho.$$

This gives $$\phi^2\rho = g^2\rho, \text{ \&c.},$$

so that by (k) we have

$$g^3 - m g^2 + m g - m = 0.$$

This must have one real root, and may have three. Suppose g_1 to be a root, then

$$\phi\rho - g_1\rho = 0,$$

and therefore, whatever be λ,

$$S\lambda\phi\rho - g_1 S\lambda\rho = 0,$$

or $$S.\rho(\phi'\lambda - g_1\lambda) = 0.$$

Thus it appears that the operator $\phi' - g_1$ cuts off from any vector λ the part which is parallel to the required value of ρ, and therefore that we have

$$\begin{aligned}\rho &\parallel MV.(\phi' - g_1)\lambda(\phi' - g_1)\mu \\ &\parallel \{m\phi^{-1} - g_1(m_2 - \phi) + g_1^2\}\zeta,\end{aligned}$$

where ζ is absolutely any vector whatever. This may be written as

$$\begin{aligned}\rho &\parallel \left\{\frac{m}{g_1} - (m_2 - g_1)\phi + \phi^2\right\}\zeta \\ &\parallel \frac{\phi^3 - m_2\phi^2 + m_1\phi - m}{\phi - g_1}\zeta.\end{aligned}$$

The same result may more easily be obtained thus.

The expression

$$(\phi^3 - m_2\phi^2 + m_1\phi - m)\rho = 0,$$

being true for all vectors whatever, may be written

$$(\phi - g_1)(\phi - g_2)(\phi - g_3)\rho = 0,$$

and it is obvious that each of these factors deprives ρ of the portion corresponding to it: i.e. $\phi - g_1$ applied to ρ cuts off the part parallel to the root of

$$(\phi - g_1)\,\sigma = 0, \ \&c., \ \&c.$$

so that the operator $(\phi - g_2)(\phi - g_3)$ when applied to a vector leaves only that part of it which is parallel to σ where

$$(\phi - g_1)\,\sigma = 0.$$

III. Thus it appears that there is always one vector, and that there may be three vectors, whose direction is unchanged by the strain. *When there are three they are perpendicular to each other, if the strain be pure.*

For, in this case, the roots of

$$M_g = 0$$

are real. Let them be such that

$$\left.\begin{array}{l}(\phi - g_1)\,\rho_1 = 0 \\ (\phi - g_2)\,\rho_2 = 0 \\ (\phi - g_3)\,\rho_3 = 0\end{array}\right\},$$

then
$$g_1 g_2 S\rho_1\rho_2 = S\phi\rho_1\phi\rho_2$$
$$= S\rho_1\phi\phi\rho_2$$

(because, by hypothesis, the strain is pure)

$$= g_2^2 S\rho_1\rho_2,$$

for
$$\phi\rho_2 = g_2\rho_2 \text{ and } \phi^2\rho_2 = g_2^2\rho_2.$$

Hence, except in the particular case of

$$g_1 = g_2,$$

we must have

$$S\rho_1\rho_2 = 0,$$

whence the proposition.

When g_1 and g_2 are equal, ρ_1 and ρ_2 are each perpendicular to ρ_3, but *any* vector in their plane satisfies

$$\phi\sigma - g_1\sigma = 0.$$

When all three roots are equal, *every* vector satisfies

$$\phi\sigma - g_1\sigma = 0.$$

IV. Thus we see that when the strain is unaccompanied by rotation the three values of g are real. But we must take care to notice that the converse does not hold. If they be real and *different*, there are three vectors at right angles to one another which are the only lines in the body whose directions remain unchanged. When two are equal, every vector parallel to a given plane, and all vectors perpendicular to it, are unchanged in direction. When all three are equal no vector has its direction changed.

V. There is, however, a peculiarity to be noticed, which distinguishes physical strain from the results of our mathematical analysis. When one or more of the values of g has a *negative* sign, we cannot interpret *physically* the result without introducing the idea of a rotation through two right angles. For we cannot conceive a pure strain which shall, as it were, pull the parts of an originally spherical portion of the body through the centre of the sphere, and so form an ellipsoid by turning a part of the body outside in.

VI. This will appear more clearly if we take the case of a rigid body, for here we must have, whatever vectors be represented by ρ and σ,

$$\left.\begin{aligned} T\phi\rho &= T\rho \\ S\rho\sigma &= S.\phi\rho\phi\sigma \end{aligned}\right\} \dots\dots\dots\dots\dots\dots (t),$$

i. e. the lengths of vectors, and their inclinations to one another, are unaltered. In this case, therefore, the strain can be nothing but a rotation. It is easy to see that the second of these equations includes the first; so that if, for variety, we take ϕ as represented in equations (a), and write

$$\rho = x\alpha + y\beta + z\gamma,$$
$$\sigma = \xi\alpha + \eta\beta + \zeta\gamma,$$

we have, for *all* values of the six scalars $x, y, z, \xi, \eta, \zeta$, the following identity;

$$
\begin{aligned}
-(x\xi+y\eta+z\zeta) &= S.(x\alpha'+y\beta'+z\gamma')(\xi\alpha'+\eta\beta'+\zeta\gamma') \\
&= \alpha'^2 x\xi+\beta'^2 y\eta+\gamma'^2 z\zeta \\
&\quad +(x\eta+y\xi)S\alpha'\beta'+(y\zeta+z\eta)S\beta'\gamma'+(z\xi+x\zeta)S\gamma'\alpha'.
\end{aligned}
$$

This necessitates

$$
\left.\begin{aligned}
\alpha'^2=\beta'^2=\gamma'^2=-1 \\
S\alpha'\beta'=S\beta'\gamma'=S\gamma'\alpha'=0
\end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (u),
$$

i. e. the vectors α', β', γ' form, like α, β, γ, a rectangular unit system. And it is evident that *any* and *every* such system satisfies the given conditions.

VII. It may be interesting to form, for this particular case, the equation giving the values of g. We have

$$
\begin{aligned}
M_g &= \frac{S.(\phi+g)\alpha\,(\phi+g)\beta\,(\phi+g)\gamma}{S.\alpha\beta\gamma} \\
&= \frac{S.(\alpha'+g\alpha)(\beta'+g\beta)(\gamma'+g\gamma)}{S.\alpha\beta\gamma} \\
&= 1-gS(\alpha\beta'\gamma'+\alpha'\beta\gamma'+\alpha'\beta'\gamma) \\
&\qquad -g^2S(\alpha\beta\gamma'+\alpha\beta'\gamma+\alpha'\beta\gamma)+g^3.
\end{aligned}
$$

Recollecting that α, β, γ; α', β', γ' are systems of rectangular unit vectors, we find that this may be written

$$
\begin{aligned}
M_g &= 1-(g+g^2)S(\alpha\alpha'+\beta\beta'+\gamma\gamma')+g^3 \\
&= (g+1)\left[g^2-g\{1+S(\alpha\alpha'+\beta\beta'+\gamma\gamma')\}+1\right].
\end{aligned}
$$

Hence the roots of

$$M_g=0$$

are in this case; first and always,

$$g_1=-1,$$

which refers to the axis about which the rotation takes place: secondly, the roots of

$$g^2-g\{1+S(\alpha\alpha'+\beta\beta'+\gamma\gamma')\}+1=0.$$

Now the roots of this equation are imaginary so long as the coefficient of the first power of g lies *between* the limits ± 2.

Also the values of the several quantities $S\alpha\alpha'$, $S\beta\beta'$, $S\gamma\gamma'$ can never exceed the limits ± 1. When the system α, β, γ coincides

with α', β', γ', the value of each of the scalars is -1, and the coefficient of the first power of g is $+2$. When two of them are equal to $+1$ and the third to -1 we have the coefficient of the first power of $g = -2$. These are the only two cases in which the three values of g are all real.

In the first, all three values of g are equal to -1, i.e.

$$\phi\rho = \rho$$

for all values of ρ, and there is no rotation whatever. In the second case there is a rotation through two right angles about the axis of the -1 value of g.

VIII. It is an exceedingly remarkable fact that, however a body may be homogeneously strained, there is always at least one vector whose direction remains unchanged. The proof is simply based on the fact that the strain-function depends on a cubic equation (with real coefficients) which must have at least one real root.

IX. As an illustration of what precedes (though one which must be approached cautiously), suppose a body to be strained so that three vectors, α'', β'', γ'' (not coplanar, and not necessarily at right angles to one another), preserve their parallelism, becoming $e_1\alpha''$, $e_2\beta''$, $e_3\gamma''$. Then we have

$$\phi\rho S.\alpha''\beta''\gamma'' = e_1\alpha'' S.\beta''\gamma''\rho + e_2\beta'' S.\gamma''\alpha''\rho + e_3\gamma'' S.\alpha''\beta''\rho.$$

By the formulæ (m, s) we have

$$m = \frac{S.\phi\alpha''\phi\beta''\phi\gamma''}{S.\alpha''\beta''\gamma''} = e_1e_2e_3,$$

$$m_1 = \frac{S(\alpha''\phi\beta''\phi\gamma'' + \beta''\phi\gamma''\phi\alpha'' + \gamma''\phi\alpha''\phi\beta'')}{S.\alpha''\beta''\gamma''} = e_2e_3 + e_3e_1 + e_1e_2,$$

$$m_2 = \frac{S(\alpha''\beta''\phi\gamma'' + \beta''\gamma''\phi\alpha'' + \gamma''\alpha''\phi\beta'')}{S.\alpha''\beta''\gamma''} = e_1 + e_2 + e_3;$$

so that we have by (k)

$$(\phi - e_1)(\phi - e_2)(\phi - e_3)\rho = 0.$$

Though the values of g are here all real, we must not rashly adopt the conclusions of (IV.), for we must remember that α'', β'', γ'' do not, like α, β, γ, necessarily form a rectangular system.

In this case we have

$$\phi'\rho S.\alpha''\beta''\gamma'' = e_1 V\beta''\gamma'' S\alpha''\rho + e_2 V\gamma''\alpha'' S\beta''\rho + e_3 V\alpha''\beta'' S\gamma''\rho.$$

So that, by (h),

$$\begin{aligned}2\epsilon &= V.(e_1\alpha'' V\beta''\gamma'' + e_2\beta'' V\gamma''\alpha'' + e_3\gamma'' V\alpha''\beta'')\\ &= (\overline{e_2 - e_3}\,\alpha'' S\beta''\gamma'' + \overline{e_3 - e_1}\,\beta'' S\gamma''\alpha'' + \overline{e_1 - e_2}\,\gamma'' S\alpha''\beta'').\end{aligned}$$

This vanishes, or the strain is pure, if either

1. $$S\alpha''\beta'' = S\beta''\gamma'' = S\gamma''\alpha'' = 0,$$

i.e. if α'', β'', γ'' are rectangular, in which case e_1, e_2, e_3 may have any values; or

2. $e_1 = e_2 = e_3$, in which case

$$\begin{aligned}\phi'\rho S.\alpha''\beta''\gamma'' &= e_1\{V\beta''\gamma'' S\alpha''\rho + V\gamma''\alpha'' S\beta''\rho + V\alpha''\beta'' S\gamma''\rho\}\\ &= e_1\rho S.\alpha''\beta''\gamma'' \text{ by (69. 14),}\end{aligned}$$

so that

$$\phi'\rho = e_1\rho = \phi\rho$$

for every vector: a general uniform dilàtation unaccompanied by change of direction.

3. $e_1 = e_2$, and α'' and β'' both perpendicular to γ''.

From what precedes it is evident that for the complete study of a strain we must endeavour to distinguish in each case between the *pure* strain and the merely *rotational* part. If a strain be capable of being decomposed into 1st a pure strain, 2nd a rotation, it is obvious that the vectors which in the altered state of the body become the axes of the strain-ellipsoid (I.) must have been originally at right angles to one another.

The equation of the strain-ellipsoid is

$$S\rho\phi^{-2}\rho = -c^2,$$

and in this it is obvious that ϕ^{-2} is self-conjugate, or at least is to be treated as such: for a non-conjugate term in $\phi^{-2}\rho$ would be (g) of the form

$$V\epsilon\rho,$$

and would therefore not appear in the equation.

Hence, as in Chap. VI, $\overline{\phi^{-2}}\rho$ is the normal to the ellipsoid at ρ,

the bar above being used to shew that the non-conjugate term has been omitted.

Now (k') $$m\phi^{-1} = m_1 - m_2\phi + \phi^2,$$

whence

$$m\phi^{-2} = \frac{m_1}{m}(m_1 - m_2\phi + \phi^2) - m_2 + \phi,$$

or

$$m^2\phi^{-2} = (m_1^2 - mm_2) - (m_1m_2 - m)\phi + m_1\phi^2.$$

Now, by (j)

$$\phi\rho = \bar{\phi}\rho + V\epsilon\rho,$$

$$\phi^2\rho = (\bar{\phi} + V\epsilon)(\bar{\phi}\rho + V\epsilon\rho)$$

$$= \bar{\phi}^2\rho + V.\epsilon\bar{\phi}\rho + \bar{\phi}V\epsilon\rho + V.\epsilon V\epsilon\rho,$$

whence $$\overline{\phi^2}\rho = \bar{\phi}^2\rho + V.\epsilon V\epsilon\rho,$$

and therefore finally

$$m^2\overline{\phi^{-2}}\rho = (m_1^2 - mm_2)\rho - (m_1m_2 - m)\bar{\phi}\rho + m_1\overline{\phi^2}\rho,$$

$$= (m_1^2 - mm_2)\rho - (m_1m_2 - m)\bar{\phi}\rho + m_1(\bar{\phi}^2\rho + V.\epsilon V\epsilon\rho),$$

which must be $= m^2h\rho,$

if ρ is an axis of the strain-ellipsoid.

We have to shew that, if ρ_1 and ρ_2 are two of the three vectors which satisfy this equation, we have not only

$$S\rho_1\rho_2 = 0,$$

but also $$S.\phi^{-1}\rho_1\phi^{-1}\rho_2 = 0.$$

By the help of the expressions above this is easily effected. But the result is much more easily obtained as an immediate consequence of a somewhat different mode of treating the question, one which we will now give:—

If q be any quaternion, the operator $q(\ \ )q^{-1}$ turns the vector, quaternion, or body operated on round an axis perpendicular to the plane of q and through an angle equal to double that of q.

The proof of this extremely important proposition is very simple; but we refer the reader to Hamilton's *Lectures*, § 282,

Elements, § 179 (1), or Tait, § 353. It is obvious that the tensor of q may be taken to be unity, i.e. q may be considered as a mere versor, because the value of its tensor does not affect that of the operator.

A very simple but important example of this proposition is given by supposing q and r to be both vectors, α and β let us say. Then

$$\alpha\beta\alpha^{-1}$$

is the result of turning β conically through 2 right angles about α, i.e. if α be the normal to a reflecting surface and β the incident ray, $-\alpha\beta\alpha^{-1}$ is the reflected ray.

Now let the strain ϕ be effected by (1) a pure strain ϖ (self-conjugate of course) followed by the rotation $q(\quad)q^{-1}$. We have, for all values of ρ,

$$\phi\rho = q(\varpi\rho)q^{-1} \dots\dots\dots\dots (v).$$

whence

$$\phi'\rho = \varpi(q^{-1}\rho q).$$

We may of course put, as in Chap. VI,

$$\varpi\rho = e_1\alpha S\alpha\rho + e_2\beta S\beta\rho + e_3\gamma S\gamma\rho,$$

where α, β, γ form a rectangular system. Hence

$$\phi\rho = e_1 q\alpha q^{-1} S\alpha\rho + e_2 q\beta q^{-1} S\beta\rho + e_3 q\gamma q^{-1} S\gamma\rho.$$

Here the axes are parallel to

$$q\alpha q^{-1},\ q\beta q^{-1},\ q\gamma q^{-1},$$

and we have

$$S.q\alpha q^{-1} q\beta q^{-1} = S.q\alpha\beta q^{-1} = S\alpha\beta = 0, \text{ \&c.}$$

So far the matter is nearly self-evident, but we now come to the important question of the *separation of the pure strain from the rotation*. By the formulæ above we see that

$$\begin{aligned}\phi'\phi\rho &= \varpi q^{-1}\phi\rho q \\ &= \varpi q^{-1}(q\varpi\rho q^{-1})q \\ &= \varpi^2\rho,\end{aligned}$$

so that we have in symbols, for the determination of ϖ, the equation

$$\phi'\phi = \varpi^2.$$

To solve this equation we employ expressions like (k). $\phi'\phi$ being a known function, let us call it ω, and form its equation as

$$\omega^3 - m_2\omega^2 + m_1\omega - m = 0.$$

Also *suppose* that the corresponding equation in ϖ is

$$\varpi^3 - g_2\varpi^2 + g_1\varpi - g = 0,$$

where g, g_1, g_2 are unknown scalars. By the help of the given relation
$$\varpi^2 = \omega,$$

we may modify this last equation as follows:

$$\varpi\omega - g_2\omega + g_1\varpi - g = 0,$$

whence
$$\varpi = \frac{g + g_2\omega}{g_1 + \omega};$$

i. e. ϖ is given definitely in terms of the known function ω, as soon as the quantities g are found. But our given equation

$$\varpi^2 = \omega$$

may now be written
$$\left(\frac{g + g_2\omega}{g_1 + \omega}\right)^2 = \omega,$$

or
$$\omega^3 - (g_2^2 - 2g_1)\,\omega^2 + (g_1^2 - 2gg_2)\,\omega - g^2 = 0.$$

As this is an equation between ω and constants it must be equivalent to that already given: so that, comparing coefficients, we have

$$\begin{aligned} g_2^2 - 2g_1 &= m_2, \\ g_1^2 - 2gg_2 &= m_1, \\ g^2 &= m; \end{aligned}$$

from which, by elimination of g and g_2, we have

$$\left(\frac{g_1^2 - m_1}{2\sqrt{m}}\right)^2 = m_2 + 2g_1.$$

The solution of the problem is therefore reduced to that of this biquadratic equation; for, when g_1 is found, g_2 is given linearly in terms of it.

It is to be observed that in the operations above we have not

been particular as to the arrangement of factors. This is due to the fact that any functions of the *same* operator are commutative in their application.

Having thus found the pure part of the strain we have at once the rotation, for (v) gives

$$\phi\varpi^{-1}\rho = q\rho q^{-1},$$

or, as it may more expressively be written,

$$\frac{\phi}{\sqrt{\phi'\phi}} = q(\ \ )q^{-1}.$$

If instead of (v) we write

$$\phi\rho = \bar{\omega}(r\rho r^{-1}) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (v'),$$

we assume that the rotation takes place first, and is succeeded by the pure strain. This form gives

$$\phi'\rho = r^{-1}(\bar{\omega}\rho)\,r,$$

and

$$\phi\phi'\rho = \bar{\omega}^2\rho,$$

whence $\bar{\omega}$ is found as above. And then (v') gives

$$\bar{\omega}^{-1}\phi = r(\ \ )r^{-1}.$$

Thus, to recapitulate, a strain ϕ is equivalent to the pure strain $\sqrt{\phi'\phi}$ followed by the rotational strain $\phi\dfrac{1}{\sqrt{\phi'\phi}}$, or to the rotational strain $\dfrac{1}{\sqrt{\phi\phi'}}\phi$ followed by the pure strain $\sqrt{\phi\phi'}$.

This leads us, as an example, *to find the condition that a given strain is rotational only*, i. e. that a quaternion q can be found such that

$$\phi = q(\ \ )q^{-1}.$$

Here we have

$$\phi' = q^{-1}(\ \ )q,$$

or

$$\phi' = \phi^{-1} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (w).$$

But

$$m\phi^{-1} = m_1 - m_2\phi + \phi^2,$$

or

$$\left.\begin{aligned} m\phi' &= m_1 - m_2\phi + \phi^2, \\ m\phi &= m_1 - m_2\phi' + \phi'^2, \end{aligned}\right\}$$

whose conjugate is (second line above)

and the elimination of ϕ' between these two equations gives

$$m\phi = m_1 - \frac{m_2}{m}(m_1 - m_2\phi + \phi^2) + \frac{1}{m^2}(m_1 - m_2\phi + \phi^2)^2,$$

i. e.

$$0 = \left| \begin{array}{l} (m^2m_1 - mm_1m_2 + m_1^2) \\ -(m^3 - mm_2^2 + 2m_1m_2)\phi \\ -(mm_2 - 2m_1 - m_2^2)\phi^2 \\ -2m_2\phi^3 \\ +\phi^4 \end{array} \right. = \left| \begin{array}{l} (m^2m_1 - mm_1m_2 + m_1^2) \\ -(m^3 - mm_2^2 + 2m_1m_2 - m)\phi \\ +(2m_1 + m_2^2 - mm_2 - m_1)\phi^2 \\ -m_2\phi^3 \end{array} \right.$$

by using the expression for ϕ^4 from the cubic in ϕ.

Now this last expression can be nothing else than the cubic in ϕ itself, else ϕ would have *two different* sets of constants in the form (k), which is absurd, as these constants, from the mode in which they are determined, can have but single values. Thus we have, by comparing coefficients,

$$\left. \begin{array}{r} m_2^2 = 2m_1 + m_2^2 - mm_2 - m_1 \\ m_1m_2 = m^3 - mm_2^2 + 2m_1m_2 - m \\ mm_2 = m^2m_1 - mm_1m_2 + m_1^2 \end{array} \right\}.$$

The first gives

$$m_1 = mm_2,$$

by the help of which the second and third each become

$$m^3 - m = 0.$$

The value

$$m = 0$$

is to be rejected, as otherwise we should have been working with non-existent terms, and m as the ratio of the volumes of two tetrahedra is positive, so that finally

$$m = 1,$$

$$m_1 = m_2,$$

and the cubic for a rotational strain is, therefore,

$$\phi^3 - m_2\phi^2 + m_2\phi - 1 = 0,$$

or

$$(\phi - 1)\{\phi^2 + (1 - m_2)\phi + 1\} = 0,$$

where m_2 is left undetermined.

By comparison with the result of (VII.) we see that in the notation there employed

$$m_2 = -S(\alpha\alpha' + \beta\beta' + \gamma\gamma').$$

The student will perhaps here require to be reminded that in the section just referred to we employed the positive sign in operators such as $\phi + g$. In the one case the coefficients in the cubic are all positive, in the other they are alternately positive and negative. The example we have given is a particularly valuable one, as it gives a glimpse of the extent to which the separation of symbols can be safely carried in dealing with these questions.

DEF. A *simple shear* is a homogeneous strain in which all planes parallel to a fixed plane are displaced in the same direction parallel to that plane, and therefore through spaces proportional to their distances from that plane.

Let α be normal to the plane, β the direction of displacement, the former being considered as an unit-vector, and the tensor of the latter being the displacement of points at unit distance from the plane.

We obviously have, by the definition,

$$S\alpha\beta = 0.$$

Now if ρ be the vector of any point, drawn from an origin in the fixed plane, the distance of the point from the plane is

$$-S\alpha\rho.$$

Hence, if σ be the vector of the point after the shear,

$$\sigma = \phi\rho = \rho - \beta S\alpha\rho.$$

This gives

$$\phi'\rho = \rho - \alpha S\beta\rho,$$

which may be written as

$$= \rho - T\beta \,.\, \alpha S \,.\, U\beta\,\rho,$$

so that the conjugate of a simple shear is another simple shear equal to the former. But the direction of displacement in each shear is perpendicular to the unaltered planes in the other.

The equation for ϕ is easily found (by calculating m, m_1, m_2 from $(m, s,)$) to be

$$\phi^3 - 3\phi^2 + 3\phi - 1 = 0.$$

Putting $\phi'\phi = \psi$, we easily find (with $b = T\beta$)

$$\psi^3 - (3 + b^2)\psi^2 + (3 + b^2)\psi - 1 = 0.$$

Solving by the process lately described, we find

$$\left(\frac{g_1^2 - 3 - b^2}{2}\right)^2 = 3 + b^2 + 2g_1.$$

If $b = 2$, this gives $g_1 = 1$, and the farther equation

$$g_1^3 + g_1^2 - 13g_1 - 21 = 0,$$

of which $g_1 = -3$ is a root, so that

$$g_1^2 - 2g_1 - 7 = 0,$$

and $$g_1 = 1 \pm 2\sqrt{2}.$$

We leave to the student the selection (by trial) of the proper root, and the formation of the complete expressions for the pure and rotational parts of the strain in this simple and yet very interesting case.

As a simple example of the case in which two of the roots of the cubic are unreal, take the vector function when the strain is equivalent to a rotation θ about the unit vector α; the others of the rectangular system being β, γ.

Here we have, obviously,

$$\phi\alpha = \alpha,$$
$$\phi\beta = \beta\cos\theta + \gamma\sin\theta,$$
$$\phi\gamma = \gamma\cos\theta - \beta\sin\theta;$$

whence at once

$$\begin{aligned} -\phi\rho &= \alpha S\alpha\rho + (\beta\cos\theta + \gamma\sin\theta)S\beta\rho + (\gamma\cos\theta - \beta\sin\theta)S\gamma\rho \\ &= (1 - \cos\theta)\alpha S\alpha\rho - \rho\cos\theta - V\alpha\rho\sin\theta. \end{aligned}$$

Forming the quantities m, m_1, m_2 as usual, we have

$$\phi^3 - (1 + 2\cos\theta)\phi^2 + (1 + 2\cos\theta)\phi - 1 = 0,$$

or $$(\phi - 1)(\phi^2 - 2\cos\theta\,\phi + 1) = 0,$$

or $$(\phi - 1)(\phi - \cos\theta - \sqrt{-1}\sin\theta)(\phi - \cos\theta + \sqrt{-1}\sin\theta) = 0.$$

Now

$$-(\phi - 1)\rho = (1 - \cos\theta)(\alpha S\alpha\rho + \rho) - \sin\theta\, V\alpha\rho,$$

$$-(\phi - \cos\theta - \sqrt{-1}\sin\theta)\rho = (1 - \cos\theta)\,\alpha S\alpha\rho + \sin\theta\,(\rho\sqrt{-1} - V\alpha\rho),$$

$$-(\phi - \cos\theta + \sqrt{-1}\sin\theta)\rho = (1 - \cos\theta)\,\alpha S\alpha\rho - \sin\theta\,(\rho\sqrt{-1} + V\alpha\rho).$$

To detect the components which are destroyed by each of these factors separately, we have, by (II.), for $(\phi - 1)$, the vector

$$(\phi^2 - 2\cos\theta\,\phi + 1)\rho = -2\alpha S\alpha\rho\,(1 - \cos\theta);$$

so that $$(\phi - 1)\alpha = 0,$$

which is, of course, true. Again

$$(\phi - 1)(\phi - \cos\theta - \sqrt{-1}\sin\theta)\rho = -\sin\theta\,(1 - \epsilon^{-\theta\sqrt{-1}})(\sqrt{-1}\,\alpha + 1)\,V\alpha\rho,$$

which we leave to the student to verify. The imaginary directions which correspond to the unreal roots are thus, in this case, parallel to the *Bivectors*

$$(\alpha \pm \sqrt{-1})\,V\alpha\rho.$$

Here, however, we reach notions which, though by no means difficult, cannot well be called elementary.

A very curious case, whose special interest however is rather mathematical than physical, is presented by the assumptions

$$\alpha' = \beta + \gamma,$$
$$\beta' = \gamma + \alpha,$$
$$\gamma' = \alpha + \beta,$$

for then $$\begin{aligned}\phi\rho &= (\beta + \gamma)\,S\alpha\rho + (\gamma + \alpha)\,S\beta\rho + (\alpha + \beta)\,S\gamma\rho \\ &= (\alpha + \beta + \gamma)\,S(\alpha + \beta + \gamma)\rho - (\alpha S\alpha\rho + \beta S\beta\rho + \gamma S\gamma\rho) \\ &= 3\delta S\delta\rho + \rho,\end{aligned}$$

where δ is a known unit vector. This function is obviously self-conjugate. Its cubic is

$$\phi^3 - 3\phi + 2 = 0 = (\phi - 1)^2(\phi + 2),$$

which might easily have been seen from the facts that

$$1\text{st}, \quad \phi\delta = -2\delta,$$
$$2\text{nd}, \quad \phi\alpha = \alpha, \text{ if } S\alpha\delta = 0.$$

The case is but slightly altered when the *signs* of α', β', γ' are changed. Then

$$\phi\rho = -3\delta S\delta\rho - \rho,$$

and the cubic is

$$\phi^3 - 3\phi - 2 = (\phi + 1)^2 (\phi - 2) = 0.$$

These are mere particular cases of extension parallel to the single axis δ. The general expression for such extension is obviously

$$\phi\rho = \rho - e\delta S\delta\rho,$$

and we have for its cubic

$$(\phi - 1)^2 \{\phi - (1 + e)\} = 0.$$

We will conclude our treatment of strains by solving the following problem: *Find the conditions which must be satisfied by a simple shear which is capable of reducing a given strain to a pure strain.*

Let ϕ be the given strain, and let the shear be, as above,

$$\psi = 1 + \beta S \,.\, \alpha,$$

then the resultant strain is

$$\psi\phi = \phi + \beta S \,.\, \alpha\phi,$$
$$= \phi + \beta S \,.\, \phi'\alpha.$$

Taking the conjugate and subtracting, we must have

$$0 = \psi\phi - \phi'\psi' = \phi - \phi' + \beta S \,.\, \phi'\alpha - \phi'\alpha S \,.\, \beta$$
$$= 2V \,.\, \epsilon - V \,.\, V\phi'\alpha\beta,$$

so that the requisite conditions are contained in the sole equation

$$2\epsilon = V\phi'\alpha\beta.$$

This gives (1) $S \,.\, \beta\epsilon = 0$,

(2) $S\phi'\alpha\epsilon = 0 = S\alpha\phi\epsilon$.

But (3) $S\alpha\beta = 0$ (by the conditions of a shear),

so that $x\alpha = V \,.\, \beta\phi\epsilon$.

Again, (4) $$2\epsilon^2 = S . \phi' \alpha \beta\epsilon = S . \alpha\phi(\beta\epsilon)$$

$$2x\epsilon^2 = S . \beta\phi\epsilon\, \phi(\beta\epsilon) = -m\beta^2\epsilon^2,$$

or $$-m\alpha = 2V . \beta^{-1}\phi\epsilon.$$

Hence we may assume any vector perpendicular to ϵ for β, and α is immediately determined.

When two of the roots of the cubic in ϕ are imaginary let us suppose the three roots to be

$$e_1,\ e_2 \pm e_3\sqrt{-1}.$$

Let β and γ be such that

$$\phi(\beta + \gamma\sqrt{-1}) = (e_2 + e_3\sqrt{-1})(\beta + \gamma\sqrt{-1}).$$

Then it is obvious that, by changing throughout the sign of the imaginary quantity, we have

$$\phi(\beta - \gamma\sqrt{-1}) = (e_2 - e_3\sqrt{-1})(\beta - \gamma\sqrt{-1}).$$

These two equations, when expanded, unite in giving by equating the real and imaginary parts the values

$$\left.\begin{aligned} \phi\beta &= e_2\beta - e_3\gamma \\ \phi\gamma &= e_2\gamma + e_3\beta \end{aligned}\right\}.$$

To find the values of α, β, γ we must, as before, operate on any vector by two of the factors of the cubic.

As an example, take the very simple case

$$\phi\rho = eVi\rho.$$

Here it is easily seen by (m, s) that $m = 0$, $m_1 = +e^2$, $m_2 = 0$,

so that $$\phi^3 + e^2\phi = 0,$$

that is $$\phi(\phi + e\sqrt{-1})(\phi - e\sqrt{-1}) = 0.$$

As operand take

$$\rho = ix + jy + kz,$$

then
$$\begin{aligned} \alpha &\parallel V(\phi + e\sqrt{-1})(\phi - e\sqrt{-1})\rho \\ &\parallel eV . (\phi + e\sqrt{-1})(ky - jz - \rho\sqrt{-1}) \\ &\parallel (-jy - kz + \rho) \\ &\parallel i. \end{aligned}$$

Again

$$\begin{aligned}\beta-\gamma\sqrt{-1} &\parallel \phi(\phi+e\sqrt{-1})\rho \\ &\parallel e\phi(ky-jz+\sqrt{-1}\,\rho) \\ &\parallel -jy-kz+\sqrt{-1}\,(ky-jz) \\ &\parallel jy+kz-\sqrt{-1}\,(jz-ky).\end{aligned}$$

With a change of sign in the imaginary part, this will represent

$$\beta+\gamma\sqrt{-1},$$

so that

$$\beta=jy+kz,$$
$$\gamma=jz-ky,$$

Thus, as the student will easily find by trial, β and γ form with α a rectangular system. But for all that the system of principal vectors of ϕ, viz.

$$\alpha,\ \beta\pm\gamma\sqrt{-1}$$

does not satisfy the conditions of rectangularity. In fact we see by the above values of β and γ that

$$S.(\beta+\gamma\sqrt{-1})(\beta-\gamma\sqrt{-1})=\beta^2+\gamma^2=-2\,(y^2+z^2).$$

It may be well to call the student's attention at this point to the fact that the tensors of these imaginary vectors vanish, for

$$T^2(\beta\pm\gamma\sqrt{-1})=-S(\beta\pm\gamma\sqrt{-1})(\beta\pm\gamma\sqrt{-1})=\gamma^2-\beta^2=0.$$

This gives, a simple example of the new and very curious modifications which our results undergo when we pass to *Bivectors;* or, more generally, to *Biquaternions.*

As a pendant to the last problem we may investigate the relation of two vector-functions whose successive application produces rotation merely.

Here

$$\phi=\psi\chi^{-1}$$

is such that by (w)

$$\phi'=\phi^{-1},$$

$$\text{i. e. } \chi'^{-1}\psi'=\chi\psi^{-1},$$

or

$$\chi'\chi=\psi'\psi=\varpi^2,$$

since each of these functions is evidently self-conjugate. This shews that the pure parts of the strains ψ and χ are the same, which is the sole condition.

One solution is, obviously,

$$\chi' = \chi^{-1}, \quad \psi' = \psi^{-1},$$

i. e. each of the two is itself a rotation; and a new proof that any number of successive rotations can be compounded into a single one may easily be given from this.

But we may also suppose either of ψ, χ, suppose the latter, to be self-conjugate, so that

$$\chi' = \chi = \bar{\chi},$$

or

$$\psi'\psi = \bar{\chi}^2,$$

which leads to previous results.

Examples to Chapter X.

1. If α, β, γ be a rectangular unit system

$$S . V\alpha\phi\alpha V\beta\phi\beta V\gamma\phi\gamma = -mS . \beta\phi'^{-1}\alpha S . \beta(\phi - \phi')\alpha,$$

and therefore vanishes if ϕ be self-conjugate. State in words the theorem expressed by its vanishing.

2. With the same supposition find the values of

$$\Sigma V . V\alpha\phi\alpha . V\beta\phi\beta \text{ and of } \Sigma S . V\alpha\phi\alpha V\beta\phi\beta.$$

Also of $$\Sigma . \alpha S\alpha\phi\alpha.$$

3. When are two simple shears commutative?

4. Expand $\dfrac{1}{1 - e\phi}$ in powers of ϕ, and reduce the result to three terms by the cubic in ϕ.

5. Shew that $$\phi' V . \phi\rho\phi^2\rho = \frac{S . \phi\rho\phi^2\rho\phi^3\rho}{S . \rho\phi\rho\phi^2\rho} V . \rho\phi\rho$$
$$= mV\rho\phi\rho.$$

6. Why cannot we expand ϕ' in terms of ϕ^0, ϕ, ϕ^2?

7. Express $V\rho\phi\rho$ in terms of ρ, $\phi\rho$, $\phi^2\rho$, and from the result find the conditions that $\phi\rho$ shall be parallel to ρ.

8. Given the coefficients of the cubic in ϕ, find those of the cubics in ϕ^2, ϕ^3, &c. ϕ^n.

9. Prove

$$\phi V.\alpha\phi'\alpha - mV.\alpha\phi'^{-1}\alpha = 0,$$

$$(\phi + m_2)\, V.\alpha\phi'\alpha = V\alpha\phi'^{2}\alpha.$$

10. If $m = \begin{vmatrix} A, & b, & c \\ a, & B, & c' \\ a', & b', & C \end{vmatrix}$ shew that $M_g = 0$ may be written as

$$\left\{g^3 \frac{d^3}{dAdBdC} + g^2\left(\frac{d^2}{dAdB} + \ldots\right) + g\left(\frac{d}{dA} + \ldots\right) + 1\right\} m = 0,$$

$$\text{or } \epsilon^{g\left(\frac{d}{dA}+\ldots\right)} m = 0.$$

11. Interpret the invariants m_1 and m_2 in connection with Homogeneous Strain.

12. The cubics in $\phi\psi$ and $\psi\phi$ are the same.

13. Find the unknown strains ϕ and χ from the equations

$$\phi + \chi = \varpi,$$
$$\phi\chi = \theta.$$

14. Shew that the value of $V(\phi\alpha\chi\alpha + \phi\beta\chi\beta + \phi\gamma\chi\gamma)$ is the same, whatever rectangular unit system is denoted by α, β, γ.

15. Find a system of simple shears whose successive application results in a pure strain.

16. Shew that, if ϕ be self-conjugate, and ξ, η two vectors, the two following equations are consequences one of the other :—

$$\frac{\xi}{S^{\frac{1}{3}}.\xi\phi\xi\phi^2\xi} = \frac{V.\eta\phi\eta}{S^{\frac{2}{3}}.\eta\phi\eta\phi^2\eta},$$

$$\frac{\eta}{S^{\frac{1}{3}}.\eta\phi\eta\phi^2\eta} = \frac{V.\xi\phi\xi}{S^{\frac{2}{3}}.\xi\phi\xi\phi^2\xi}.$$

From either of them we obtain the equation :

$$S\phi\xi\phi\eta = S^{\frac{1}{3}}.\xi\phi\xi\phi^2\xi\, S^{\frac{1}{3}}.\eta\phi\eta\phi^2\eta.$$

17. Shew that in general any self-conjugate linear and vector function may be expressed in terms of two given ones, the expression involving terms of the second order.

Shew also that we may write

$$\phi + z = a(\varpi + x)^2 + b(\varpi + x)(\omega + y) + c(\omega + y)^2,$$

where a, b, c, x, y, z are scalars, and ϖ, ω the given functions. What character of generality is necessary in ϖ and ω? How is the solution affected by non-self-conjugation in one or both?

18. Solve the equations:

(a) $V.\alpha\rho\beta = V.\alpha\gamma\beta$,

(b) $\alpha\rho + \rho\beta = \gamma$,

(c) $\rho + \alpha\rho\beta = \alpha\beta$,

(d) $\alpha\rho\alpha^{-1} + \beta\rho\beta^{-1} = \gamma\rho\gamma^{-1}$,

(e) $\alpha\rho\beta\rho = \rho\alpha\rho\beta$.

APPENDIX.

We have thought it would be acceptable to many students if we should give as an Appendix a brief, and in some cases even a detailed, solution of the most important and most difficult of the Additional Examples. In doing so, we would add as a word of advice, that our solutions be employed simply for the purpose of comparison with those which shall occur to the student himself.

Chap. II.

Ex. 4. If $AB=\alpha$, $BC=\beta$, $AP=m\alpha$, $AP'=m'\alpha$, $BQ=m\beta$, &c.; then

$$AE=AP+xPQ=AP'+x'P'Q'$$

gives $\quad m\alpha+x\{(1-m)\,\alpha+m\beta\}=m'\alpha+x'\{(1-m')\,\alpha+m'\beta\},$

whence $\qquad x=m'$, and $PE=m'PQ$.

Ex. 6. $ABCD$ is a quadrilateral; $AB=\alpha$, $AC=\beta$, $AD=\gamma$, $AP=m\alpha$, $BQ=m\,(\beta-\alpha)$, &c.

The condition $\qquad PQ+RS=0$

gives $\qquad (1-m)\,\alpha+m\,(\beta-\alpha)+(1-m)\,(\gamma-\beta)-m\gamma=0,$

or $\qquad (1-2m)\,(\alpha-\beta+\gamma)=0\,;$

an equation which is satisfied either when $1-2m=0$, or when $\alpha-\beta+\gamma=0$.

The former solution is Ex. 5; the latter gives $ABCD$ a parallelogram.

Ex. 10. Let a, b, c be the points in which the bisectors of the exterior angles at A, B, C meet the opposite sides. Let unit

vectors along BC, CA, AB be α, β, γ; then with the usual notation we have

$$a\alpha + b\beta + c\gamma = 0 \quad \text{.......................(1).}$$

Now $$A\alpha = x(\beta + \gamma) = -b\beta + y(b\beta + c\gamma)$$

gives $$x = \frac{bc}{b-c};$$

and $$A\alpha = \frac{bc}{b-c}(\beta + \gamma).$$

Similarly $$Bb = \frac{ca}{c-a}(\gamma + \alpha),$$

$$Cc = \frac{ab}{a-b}(\alpha + \beta),$$

therefore $$Ab = -\frac{bc}{c-a}\beta, \text{ (by 1)},$$

$$Ac = -\frac{bc}{a-b}\gamma.$$

Hence $$(b-c)A\alpha + (c-a)Ab + (a-b)Ac = 0,$$

and also $$(b-c) + (c-a) + (a-b) = 0,$$

therefore (Art. 13) a, b, c are in a straight line.

COR. $$ba : ca :: b-a : c-a.$$

Ex. 12. If the figure of Ex. 11, Art. 23, be supposed to represent a parallelepiped; then, with the notation of that example, the vector from O to the middle point of OG is $\frac{1}{2}(\alpha + \beta + \delta)$, which is the same as the vector to the middle point of AF, viz.

$$\alpha + \frac{1}{2}(\beta + \delta - \alpha).$$

Ex. 13. With the figure and notation of Art. 31, the former part of the enunciation is proved by the equation

$$\frac{\alpha + \beta + \gamma}{4} = \frac{1}{4}\left(\frac{\alpha + \beta + \gamma}{3} + \frac{\alpha + \beta}{3} + \frac{\beta + \gamma}{3} + \frac{\gamma + \alpha}{3}\right).$$

Also, if the edges AB, BC, CA be bisected in c, a, b, the mean point of the tetrahedron $Oabc$ is evidently

$$\frac{1}{4}\left(\frac{\alpha+\beta}{2}+\frac{\beta+\gamma}{2}+\frac{\gamma+\alpha}{2}\right),$$

which proves the latter part of the enunciation.

Ex. 14. Here we have to do with nothing but the triangles on each side of OD.

If $OQ=\alpha$, $QA=p\alpha$, $AP=\beta$, $PD=q\beta$;

$$TO=xOD=TQ-OQ=yQP-OQ$$

gives $$x=\frac{1}{pq-1}.$$

Similarly, if $OS=\alpha'$, $SB=p'\alpha'$, $BR=\beta'$, $RD=q'\beta'$;

$$T'O=x'OD$$

gives $$x'=\frac{1}{p'q'-1}.$$

But the data are $\frac{1}{q}=\frac{m}{p'}$, $p=mq'$; hence

$$pq=p'q', \text{ and } x=x';$$

therefore T' coincides with T.

Ex. 15. If $AB=\alpha$, $AC=\beta$, $MN=p\alpha$, $PQ=q\beta$, $RS=r(\beta-\alpha)$, we shall have, by making $AO=AP+PO=AR+RO$,

$$(1-q)\alpha+(1-p)\beta=r\alpha+(1-p)(\beta-\alpha);$$

therefore $$p+q+r=2.$$

Ex. 17. Let $RA=\alpha$, $RB=\beta$, $AP=m\alpha$, $AD=p\alpha+q\beta$; then

$$PD=p\alpha+q\beta-m\alpha,$$

and $RS=RP+PS=RQ+QS$ gives

$$(1+m)\alpha+x(p\alpha+q\beta-m\alpha)=(1+m)\beta+y(p\alpha+q\beta-m\beta),$$

whence $$x=\frac{1+m}{m},$$

and $$RS=\frac{1+m}{m}(p\alpha+q\beta)=\frac{1+m}{m}AD.$$

CHAP. III.

Ex. 5. Let $ABCD$ be the quadrilateral; DA, DB, DC, α, β, γ respectively.

Now $$\beta(\gamma-\alpha)+(\gamma-\alpha)\beta=\gamma(\beta-\alpha)+(\beta-\alpha)\gamma$$
$$+\alpha(\gamma-\beta)+(\gamma-\beta)\alpha.$$

Taking scalars, and applying 22. 3, there results,

$$S\beta(\gamma-\alpha)=S\gamma(\beta-\alpha)+S\alpha(\gamma-\beta),$$

which is the proposition.

Ex. 6. If α, β, γ be the vectors OA, OB, OC corresponding to the edges a, b, c; we have

$$\begin{aligned} V(CA\,.\,CB) &= V(\alpha-\gamma)(\beta-\gamma) \\ &= V(\alpha\beta+\beta\gamma+\gamma\alpha) \\ &= abk+bci+caj, \end{aligned}$$

the negative square of which is the proposition given.

Ex. 7. If $S\alpha(\beta-\gamma)=0$ and $S\beta(\alpha-\gamma)=0$, then, by subtraction, will $S\gamma(\alpha-\beta)=0$.

Ex. 8. If $\alpha^2=(\beta-\gamma)^2$, $\beta^2=(\gamma-\alpha)^2$, $\gamma^2=(\alpha-\beta)^2$; then will

$$S\left(\frac{\beta+\gamma}{2}-\frac{\alpha}{2}\right)\left(\frac{\gamma+\alpha}{2}-\frac{\beta}{2}\right)=0, \text{ \&c. \&c.},$$

for these are the same equations in another form; and they prove that the corresponding vectors are at right angles to one another.

Ex. 9. If OA, OB, OC, OD are α, β, γ, δ;

$$\begin{aligned} \text{triangle } DAB : DAC &:: \text{tetrahedron } ODAB : ODAC \\ &:: S\alpha\beta\delta : S\alpha\gamma\delta \\ &:: \text{triangle } OAB : OAC, \end{aligned}$$

because the angles which δ makes with the planes OAB, OAC are equal.

Chap. IV.

Ex. 1. Let O be the middle point of the common perpendicular to the two given lines; α, $-\alpha$, the vectors from 0 to those lines, unit vectors along which are β, γ; ρ the vector to a point P in a line QR which joins the given lines; P being such that $RP = mPQ$; therefore

$$\rho + \alpha - y\gamma = m(\alpha + x\beta - \rho).$$

Now since α is perpendicular to both β and γ, the equation gives $(1 + m) S\alpha\rho = (m - 1)\alpha^2$; a plane.

Ex. 2. Retaining what is necessary of the notation of the last example, let $OS = \delta$.

If PR perpendicular on γ meet β in Q, we have

$$-\alpha + y\gamma + RP = \rho, \text{ which gives } y\gamma^2 = S\gamma\rho;$$

$$RQ = 2\alpha + x\beta - y\gamma, \text{ which gives } y\gamma^2 = xS\beta\gamma;$$

and $SP^2 = e^2PQ^2$ gives

$$(\rho - \delta)^2 = e^2(\alpha + x\beta - \rho)^2$$

$$= e^2\left(\alpha + \frac{S\gamma\rho}{S\beta\gamma}\beta - \rho\right)^2,$$

which being of the second degree in ρ shews that the locus is a surface of the second order. See Chap. VI.

Ex. 3. The equation of the plane is

$$S\gamma\rho = a,$$

which, being substituted in the equation of the surface, gives what is obviously the equation of a circle.

Ex. 4. With the notation of Ex. 1, let δ, δ' be the perpendiculars on the lines,

then $\qquad \rho + \delta = \alpha + x\beta$ gives $V\beta\delta = -V\beta(\rho - \alpha)$,

and the condition given may be written

$$V^2\beta\delta = e^2V^2\gamma\delta';$$

$$\therefore\ V^2\beta(\rho-\alpha)=e^2V^2\gamma(\rho+\alpha).$$

Now (22. 9)

$$V^2\beta(\rho-\alpha)=-\beta^2(\rho-\alpha)^2+S^2\beta(\rho-\alpha),$$

whence $\rho^2-2S\alpha\rho+\alpha^2+S^2\beta\rho=e^2(\rho^2+2S\alpha\rho+\alpha^2+S^2\gamma\rho),$

a surface of the second order.

Ex. 6. $S\rho(\beta+\gamma)=c$, a plane perpendicular to the line which bisects the angle which parallels to the given lines drawn through O make with one another.

Ex. 7. α, β the vectors to the given points A, B,

$$S\gamma\rho=a,\ S\delta\rho=b$$

the equations of the planes, γ, δ being unit vectors.

$x\gamma$, $y\delta$ the vector perpendiculars from A on the planes, then

$$x=S\alpha\gamma-a,\ y=S\alpha\delta-b,$$

$$\therefore\ x+y=S\alpha(\gamma+\delta)-(a+b)\ \ldots\ldots\ldots\ldots\ (1).$$

Hence by the question

$$S\alpha(\gamma+\delta)=S\beta(\gamma+\delta)$$

or

$$S(\beta-\alpha)(\gamma+\delta)=0\ \ldots\ldots\ldots\ldots\ldots\ (2).$$

Now equation (1) will give the sum of the perpendiculars on the planes from any other point in the line AB by simply writing $\alpha+z(\beta-\alpha)$ in place of α; and from equation (2) this will produce no change.

Ex. 8. If β' be the vector to C, equation (2) of the last example gives

$$S(\beta-\alpha)(\gamma+\delta)=0,\ S(\beta'-\alpha)(\gamma+\delta)=0.$$

Now the sum of the perpendiculars from any other point in the plane will be found from equation (1) by writing

$$\alpha+z(\beta-\alpha)+z'(\beta'-\alpha)$$

in place of α. Hence the proposition.

Ex. 10. Tait's *Quaternions*, Art. 213.

Ex. 11. Let α, β, γ, δ be the vectors OA, OB, OC, OD; then (34. 5, Cor.)

$$\delta = S.\alpha\beta\gamma.(V\alpha\beta + V\beta\gamma + V\gamma\alpha)^{-1}$$
$$= \frac{abc\,(bci + caj + abk)}{(ab)^2 + (bc)^2 + (ca)^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Now

$$\begin{aligned}
&\text{triangle } ABD : \text{triangle } ABC \\
&\quad :: \text{tetrahedron } OABD : \text{tetrahedron } OABC \\
&\quad :: S.\alpha\beta\delta : S.\alpha\beta\gamma \\
&\quad :: S.abij\delta : S.abcijk \\
&\quad :: (ab)^2 : (ab)^2 + (bc)^2 + (ca)^2 \\
&\quad :: (\text{triangle } AOB)^2 : (\text{triangle } ABC)^2.
\end{aligned}$$

(Chap. III., Additional Ex. 6.)

Ex. 12. This is merely the equation

$$\rho = \alpha t + \frac{\beta}{t},$$

with t eliminated by taking the product of $V\alpha\rho$, $V\beta\rho$. (See 55. 3.)

Chap. V.

Ex. 3. Let a, a' be the radii of the circles; α, ρ the vectors from the centre of one of them to that of the other, and to the point whose locus is required; then

$$\frac{T\rho}{a} = \frac{T(\rho - \alpha)}{a'}.$$

Ex. 7. This is the polar reciprocal of Ex. 3, Art. 40.

Ex. 8. Let A be the origin, $AB = \beta$, $AC = \gamma$, the vector to the centre α: then

$$\begin{aligned}
-V(AB.BC.CA) &= V.\beta(\gamma - \beta)\gamma \\
&= \gamma^2\beta - \beta^2\gamma \\
&= 2\beta S\alpha\gamma - 2\gamma S\alpha\beta \text{ from the circle;}
\end{aligned}$$

$$\therefore S.\alpha V(AB.BC.CA) = 0.$$

Ex. 9. Tait, Art. 222.

Ex. 10. Tait, Art. 221.

Ex. 11. Tait. Art. 223.

Ex. 12. Tait, Art. 232.

CHAP. VI.

Ex. 1. Let δ be the vector to the given point, π the vector to the point of bisection of a chord, β a vector parallel to the chord, all measured from the centre; then

$$\delta = \pi + x\beta,$$

$$S\pi\phi\delta = S\pi\phi\pi \qquad \text{(48)};$$

from which by making

$$\pi = \rho + \frac{1}{2}\delta,$$

we get

$$S\rho\phi\rho = \frac{1}{4} S\delta\phi\delta,$$

an ellipse whose centre is at the point of bisection of the line which joins the given point with the centre of the given ellipse.

Ex. 2. Let $2b$ be the shortest distance between the given lines; θ their angle of inclination; $2a$ the line of constant length; then as in Ex. 2, Chap. IV.,

$$-4a^2 = (2\alpha + x\beta - y\gamma)^2,$$

$$2\rho = x\beta + y\gamma;$$

the former gives

$$x^2 + y^2 - 2xy\cos\theta = 4(a^2 - b^2) \qquad \text{(1)},$$

the latter

$$4\rho = (x+y)(\beta+\gamma) + (x-y)(\beta-\gamma),$$

which, since $\beta+\gamma$, $\beta-\gamma$ are vectors bisecting the angles between the lines and therefore at right angles to one another, is an equation of the form of that in Art. 55. 2; whilst equation (1) satisfies the condition

$$(x+y)^2 + m(x-y)^2 = c,$$

which is requisite for an ellipse.

Ex. 3. Let α be a vector semi-diameter, parallel to a chord through O; δ the vector to O: then

$$\rho = \delta + x\alpha$$

gives $$S\delta\phi\delta + 2xS\delta\phi\alpha + x^2 S\alpha\phi\alpha = 1,$$

which, since $$S\alpha\phi\alpha = 1,$$

shews that the product of the two values of x is constant; hence the rectangle by the segments of the chord varies as α^2, which is the proposition.

Ex. 4. With the usual notation, let CE, CE' be semi-diameters parallel to DP, $D'P$, and let their vectors be $m(\alpha - \beta)$, $n(\alpha + \beta)$; then since P, D, E, E' are points in the ellipse,

$$m^2 S(\alpha - \beta)\,\phi\,(\alpha - \beta) = 1,$$

$$\therefore\ 2m^2 = 1. \quad \text{Similarly } 2n^2 = 1,\ m = n,$$

$$\begin{aligned} \text{and}\ DP : D'P &:: T(\alpha - \beta) : T(\alpha + \beta) \\ &:: Tm(\alpha - \beta) : Tn(\alpha + \beta) \\ &:: CE : CE'. \end{aligned}$$

Cor. Since $m = \dfrac{1}{\sqrt{2}}$, $CE : DP :: 1 : \sqrt{2}$.

Ex. 5. Put $n\alpha'$, $n\rho'$ in place of α, ρ in equation (1), Art. 43.

Ex. 6, 7. With everything as in Ex. 4, CE, CE' being now semi-diameters in the direction of diagonals of the parallelogram,

$$\begin{aligned} SCE\phi CE' &= \frac{1}{2} S(\alpha - \beta)\,\phi\,(\alpha + \beta) \\ &= 0; \end{aligned}$$

hence CE, CE' are conjugate.

Ex. 8. $S(\alpha + \beta)\,\phi\,(\alpha + \beta) = 2$ gives an ellipse, whose equation is

$$S\rho\phi'\rho = 1, \text{ where } \phi' = \frac{\phi}{2};$$

hence the diameters of the locus are to those of the given ellipse

$$:: \sqrt{2} : 1.$$

Ex. 9. If γ be a unit vector to which the lines are parallel, ρ, ρ' points in which the lines cut the ellipse,

$$\rho = ai + m\gamma,\ \rho' = bj + n\gamma,$$

and

$$S\rho\phi\rho = 1 \text{ gives}$$

Similarly

$$\left.\begin{array}{l} 2aSi\phi\gamma + mS\gamma\phi\gamma = 0 \\ 2bSj\phi\gamma + nS\gamma\phi\gamma = 0 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Now,

$$\begin{aligned} S\rho\phi\rho' &= an\,Si\phi\gamma + bmSj\phi\gamma + mnS\gamma\phi\gamma \\ &= 0, \text{ by equations (1)}; \end{aligned}$$

$$\therefore\ \rho,\ \rho' \text{ are conjugate.}$$

Cor. The same demonstration applies when the diameters from whose extremities parallels are drawn, are any conjugate diameters whatever, i, j being parallel to those diameters.

Ex. 10. Let CP, CP' be any two semi-diameters, their vectors being α, α'; PQ the semi-ordinate to CP'; $CQ = n\alpha'$; then

$$S(PQ\,.\,\phi\alpha') = 0$$

gives

$$S(\alpha - n\alpha')\,\phi\alpha' = 0,$$

$$\therefore\ n = S\alpha\phi\alpha'.$$

Now the area of the triangle QCP is proportional to

$$V(CP\,.\,CQ),$$

i.e. to $nV\alpha\alpha'$ or to

$$S\alpha\phi\alpha'\,.\,V\alpha\alpha',$$

which, being symmetrical in α, α', proves the proposition.

Ex. 11. If the tangent at P' meet CP produced in T,

$$CT = m\alpha;$$

then, since $P'T$ is perpendicular to $\phi\alpha'$,

$$S(CT - \alpha')\,\phi\alpha' = 0.$$

$$\therefore\ m = \frac{1}{S\alpha\phi\alpha'};$$

and area $P'CT$ is proportional to $V(CP'\,.\,CT)$, i.e. to $\dfrac{V\alpha\alpha'}{S\alpha\phi\alpha'}$, which is symmetrical in α, α'.

Ex. 12. Let α, β be the vector semi-diameters of the larger ellipse; C the centre; O the centre of the smaller ellipse, whose equation is

$$S\rho\phi\rho = c,$$

γ a vector along PQR; then

$$OQ = \frac{\alpha+\beta}{2} + x\gamma;$$

$$\therefore\ S\left(\frac{\alpha+\beta}{2} + x\gamma\right)\phi\left(\frac{\alpha+\beta}{2} + x\gamma\right) = c$$

$$= S\frac{\alpha+\beta}{2}\phi\frac{\alpha+\beta}{2},$$

$$\therefore\ x = -\frac{S(\alpha+\beta)\phi\gamma}{S\gamma\phi\gamma};$$

and since
$$CQ = \alpha+\beta+x\gamma,$$

$$S(CQ\phi\gamma) = 0;$$

hence PR is conjugate to CQ, and therefore bisected at Q.

Ex. 13. This is simply a combination of 49. 2 and 49. 1.

CHAP. VII.

Ex. 3. The equation of the circle is

$$\left(\rho - \frac{\alpha}{2}\right)^2 = \frac{9}{16}\alpha^2,$$

which by 52. 1 gives

$$(\alpha^2 - S\alpha\rho)^2 - \alpha^2 S\alpha\rho = \frac{5}{16}\alpha^4,$$

$$\therefore\ S\alpha\rho = \frac{\alpha^2}{4},$$

which (52. 11) is the proposition.

Ex. 5. If O be the centre of the circle, Q a point at which it meets the tangent at A; then, with the notation of 55. 1,

$$QO^2 = \{a\alpha + \frac{1}{2}(\rho - a\alpha) - z\beta\}^2 = \frac{1}{4}(\rho - a\alpha)^2,$$

$$\therefore\ z^2\beta^2 - zS\beta\rho + aS\alpha\rho = 0,$$

$$\text{i.e.}\quad z^2 - zy + \frac{y^2}{4} = 0,$$

which gives two equal values of z; hence the proposition.

Ex. 6. With any point as origin, let β, γ be the vectors to the two given points, π the vector to the focus of one of the parabolas. Write $a\alpha$ in place of α in equation (1), Art. 52, α being a unit vector;

then
$$-(\beta-\pi)^2 = \{a + S\alpha(\beta-\pi)\}^2 \dots\dots\dots\dots\dots \quad (1)$$
$$-(\gamma-\pi)^2 = \{a + S\alpha(\gamma-\pi)\}^2,$$

whence, by subtraction,

$$\beta^2 - \gamma^2 - 2S\pi(\beta-\gamma) = -S\alpha(\beta-\gamma)\{2a + S\alpha(\beta-\gamma) - 2S\alpha\pi\},$$

which gives a by a simple equation in π; and then equation (1) becomes a quadratic in π.

Ex. 8. If two tangents meet at T, it is easy, as in Ex. 5, Art. 55, with the notation available for the focus, to find

$$ST = \frac{yy'}{4a}\alpha + \frac{y+y'}{2}\beta - a\alpha,$$

$$ST' = \frac{yy''}{4a}\alpha + \frac{y+y''}{2}\beta - a\alpha,$$

and $S(ST\,.\,ST') = 0$ will follow at once, from the fact that

$$y'y'' + 4a^2 = 0.$$

Ex. 9. Let P be the point of contact, PQ the chord, TEF the line parallel to the axis cutting the curve in E; E the origin;

$$EP = \frac{t^2}{2}\alpha + t\beta,\quad ET = -\frac{t^2}{2}\alpha,$$

$$EP = EF + FP = y\alpha + z\left\{\frac{t^2 - t'^2}{2}\alpha + (t-t')\beta\right\};$$

whence
$$z = \frac{t}{t-t'},\quad y = -\frac{tt'}{2}.$$

$$\therefore\ PF : FQ :: t : t'$$
$$:: \frac{t^2}{2} : \frac{tt'}{2}$$
$$:: TE : EF.$$

Ex. 10. This is evident from equation (1) Art. 52.

Ex. 11. With the notation of Art. 52, let

$$SQ = xPS = -x\rho,\ Ay = -x'AP = x'\left(\frac{a}{2} - \rho\right),$$
$$\therefore\ x'(\alpha \dot{-} 2\rho) = \alpha + \delta,$$
$$x'(\alpha^2 - 2S\alpha\rho) = \alpha^2.$$

But $\rho, -x\rho$ being vectors to the parabola, equation (1), Art. 52, gives

$$x^2(\alpha^2 - S\alpha\rho)^2 = (\alpha^2 + xS\alpha\rho)^2,$$
$$\therefore\ x(\alpha^2 - S\alpha\rho) = \alpha^2 + xS\alpha\rho,$$
$$x(\alpha^2 - 2S\alpha\rho) = \alpha^2,$$
$$\therefore\ x = x',$$

and the proposition is true (Euc. VI. 2).

Ex. 14. Tait, Art. 43, Cor. 2.

Ex. 15.

$$CP = \alpha t + \frac{\beta}{t} \text{ gives } CT = 2\alpha t,$$
$$CQ = 2\alpha t + x\beta = \alpha t' + \frac{\beta}{t'} = 2\alpha t + \frac{\beta}{2t},$$
$$\therefore\ PQ = \alpha t - \frac{\beta}{2t},$$

so that the equation of $RQPR'$ is

$$\rho = \alpha t + \frac{\beta}{t} + x\left(\alpha t - \frac{\beta}{2t}\right),$$

whence for R and R' the values of x are 2 and -1; therefore

$$CR = 3\alpha t,\quad CR' = \frac{3}{2}\frac{\beta}{t},$$

$$QR = \alpha t - \frac{\beta}{2t} = PQ = \frac{1}{3} RR'.$$

Ex. 16. If $CR = a\alpha$; $\alpha + m\beta$, $\alpha - m\beta$ vectors parallel to the given conjugate diameters,

$$CP = a\alpha + x(\alpha + m\beta) = \alpha t + \frac{\beta}{t},$$

$$CD = a\alpha + x'(\alpha - m\beta) = \alpha t' - \frac{\beta}{t'},$$

give $t = t'$; therefore CP, CD are conjugate.

Ex. 18. Adopting the figure and notation of Ex. 2 of the hyperbola, Art. 55, we have

$$CR = 2Xt\alpha, \quad Cr = 2X\frac{\beta}{t};$$

therefore $$QR = (X - Y)\left(t\alpha - \frac{\beta}{t}\right),$$

$$rQ = (X + Y)\left(t\alpha - \frac{\beta}{t}\right),$$

and $$rQ \, . \, QR = (X^2 - Y^2)\left(t\alpha - \frac{\beta}{t}\right)^2$$

$$= PO^2, \text{ since } X^2 - Y^2 = 1.$$

As an example of combining not merely the forms but the results of the Cartesian Geometry with Quaternions, we will add one more example.

If CP, CD; CP', CD' be two pairs of conjugate semi-diameters of an ellipse, PD' will be parallel to $P'D$.

Let CP, CP' be denoted, as in Art. 55. 2, by $x\alpha + y\beta$, $x'\alpha + y'\beta$ respectively; then CD, CD' will be represented by

$$-\frac{a}{b} y\alpha + \frac{b}{a} x\beta, \quad -\frac{a}{b} y'\alpha + \frac{b}{a} x'\beta,$$

with the conditions

$$a^2y^2 + b^2x^2 = a^2b^2, \quad a^2y'^2 + b^2x'^2 = a^2b^2 \dots\dots\dots\dots (1).$$

Now vector $D'P = \left(x + \frac{a}{b} y'\right)\alpha + \left(y - \frac{b}{a} x'\right)\beta,$

$$DP' = \left(x' + \frac{a}{b}\,y\right)\alpha + \left(y' - \frac{b}{a}\,x\right)\beta.$$

But equations (1) give, by subtraction,

$$x + \frac{a}{b}y' : y - \frac{b}{a}x' :: x' + \frac{a}{b}y : y' - \frac{b}{a}x;$$

therefore $D'P$ is a multiple of DP' and consequently parallel to it.

COR. $PD' : P'D :: ay' + bx : ay + bx'.$

CHAP. VIII.

Ex. 1. With the notation of Additional Ex. 1, Chap. IV., the perpendiculars are

$$\rho - \alpha - x\beta, \quad \rho + \alpha - y\gamma,$$

so that $$S\beta\rho = x\beta^2, \quad S\gamma\rho = y\gamma^2;$$

and by the question,

$$(\rho - \alpha - \beta^{-1}S\beta\rho)^2 = e^2(\rho + \alpha - \gamma^{-1}S\gamma\rho)^2,$$

a surface of the second order in ρ.

Ex. 3. The equations $S\rho\phi\rho = 1$, $S\pi\phi\rho = 1$, with the condition $\pi = x\phi\rho$, give

$$\frac{1}{x^2}S\pi\phi^{-1}\pi = 1, \quad \frac{\pi^2}{x} = 1 \text{ respectively,}$$

therefore $$S\pi\phi^{-1}\pi = \pi^4,$$

whence the Cartesian equation.

Ex. 4. If α, β, γ are the vector radii,

$$\frac{S\alpha\phi\alpha}{(T\alpha)^2} = \frac{(SiU\alpha)^2}{a^2} + \frac{(SjU\alpha)^2}{b^2} + \frac{(SkU\alpha)^2}{c^2},$$

&c. = &c.

Adding and observing that $S\alpha\phi\alpha = 1$, &c., there results

$$\frac{1}{(T\alpha)^2} + \frac{1}{(T\beta)^2} + \frac{1}{(T\gamma)^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

Ex. 5. As in Ex. 8, Art. 64,

$$\frac{1}{Oy_1^2} = -(\phi a)^2,$$

and if vector $OQ_1 = x\phi a$, the ellipsoid gives

$$x^2 S\phi a\phi^2 a = 1.$$

Now
$$\frac{1}{Oy_1^2 . OQ_1^2} = \frac{1}{x^2} = S\phi a\phi^2 a$$
$$= \frac{(Sia)^2}{a^6} + \frac{(Sja)^2}{b^6} + \frac{(Ska)^2}{c^6},$$

and, since

$$(Sia)^2 + (Si\beta)^2 + (Si\gamma)^2 = a^2$$

(Ex. 7, Art. 64), the result required is obtained by simply adding.

Ex. 6. Let pk be the vector distance from the origin, of the plane parallel to xy, π a point in it; then $Sk(\pi - pk) = 0$ gives $S\pi k =$ const.

Now $S\rho\phi\pi = 1$ is the equation of the plane of contact, and if zk be the point in which this plane cuts the axis of z, $zSk\phi\pi = 1$, i.e. $zS\pi\phi k = 1$, gives z.

Now ϕk is a multiple of k, and since $S\pi k$ is constant, z is constant.

Ex. 7. The equations of the ellipsoids

$$S\rho\phi\rho = 1, \; S(\rho - a)\,\phi\,(\rho - a) = 1,$$

give $S\rho\phi a =$ const. as the plane of contact.

Ex. 8. If pa be the vector to the point in the line OA; the equation of its polar plane is $Spa\phi\rho = 1$; and the square of the reciprocal of the perpendicular from the centre on this plane is $-p^2(\phi a)^2$. Hence the conclusion by Ex. 8, Art. 64.

Ex. 9. Let ρ be the vector to P; a, β, γ vector radii parallel to the chords; then

$$\rho + xa, \; \rho + y\beta, \; \rho + z\gamma,$$

will be the vectors to A, B, C; and since P, A, B, C are points in the ellipsoid

$$S\rho\phi\rho = 1,\ 2S\rho\phi\alpha + x = 0,\ 2S\rho\phi\beta + y = 0,$$
$$2S\rho\phi\gamma + z = 0.$$

The equation of the plane ABC is (34. 5)

$$S\,.\,(\pi - \rho)\,(xy\alpha\beta + yz\beta\gamma + zx\gamma\alpha) = xyzS\,.\,\alpha\beta\gamma,$$

and since α, β, γ are at right angles to one another,

$$\alpha\beta = -\frac{\gamma}{(T\gamma)^2}\,S\,.\,\alpha\beta\gamma,\ \&c.$$

therefore the equation of the plane ABC becomes

$$S\,.\,(\pi - \rho)\left\{\frac{1}{(T\gamma)^2}\,.\,\frac{\gamma}{S\rho\phi\gamma} + \frac{1}{(T\alpha)^2}\,.\,\frac{\alpha}{S\rho\phi\alpha} + \frac{1}{(T\beta)^2}\,.\,\frac{\beta}{S\rho\phi\beta}\right\} = 2,$$

which is satisfied by

$$\pi - \rho = m\phi\rho,$$

where

$$m\left\{\frac{1}{(T\alpha)^2} + \frac{1}{(T\beta)^2} + \frac{1}{(T\gamma)^2}\right\} = 2\,;$$

and therefore Ex. 4 above gives

$$m = \frac{2}{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}\,.$$

CHAP. IX.

Ex. 2 and 3. Employ formula 11.

Ex. 5. Since

$$\alpha^2\beta^2\gamma^2 = \alpha\beta\gamma\,.\,\gamma\beta\alpha,$$

formulæ 4 and 6 give the required result.

Ex. 6. Apply formula 10 to Ex. 5.

Ex. 8.
$$\begin{aligned}(\alpha\beta\gamma)^2 &= \alpha\beta\gamma\,.\,\alpha\beta\gamma = \alpha\beta\gamma\,(S\,.\,\alpha\beta\gamma + V\,.\,\alpha\beta\gamma)\\ &= \alpha\beta\gamma\,(S\,.\,\alpha\beta\gamma + V\,.\,\gamma\beta\alpha)\\ &= \alpha\beta\gamma\,(\gamma\beta\alpha + 2S\,.\,\alpha\beta\gamma)\\ &= \alpha^2\beta^2\gamma^2 + 2\alpha\beta\gamma S\,.\,\alpha\beta\gamma.\end{aligned}$$

Ex. 9. Formula 10 gives the vector of the product of three vectors α, β, γ, under the form $\alpha' - \beta' + \gamma'$ where $\alpha' = \alpha S\beta\gamma$, &c.

Hence the required scalar may be written

$$S.(\alpha' - \beta' + \gamma')(\alpha' + \beta' - \gamma')(-\alpha' + \beta' + \gamma');$$

and as the scalar part of this product is that which involves all of the three vectors α', β', γ' we have exactly as in the demonstration of formula 5,

$$\begin{aligned} &S(V\alpha\beta\gamma V\beta\gamma\alpha V\gamma\alpha\beta) \\ &= S.\begin{vmatrix} \alpha', -\beta', \gamma' \\ \alpha', \beta', -\gamma' \\ -\alpha', \beta', \gamma' \end{vmatrix} \\ &= 4S.\alpha'\beta'\gamma'. \end{aligned}$$

10. The scalar part, by formula 16, is reduced to

$$S\alpha\delta S\beta\gamma - S\alpha\gamma S\beta\delta - S\alpha\delta S\beta\gamma + S\alpha\beta S\gamma\delta + S\alpha\gamma S\beta\gamma - S\alpha\beta S\gamma\delta,$$

which is identically 0.

The vector part, by formula 12, is

$$\alpha S.\gamma\delta\beta - \beta S.\gamma\delta\alpha + \alpha S.\delta\beta\gamma - \gamma S.\delta\beta\alpha + \alpha S.\beta\gamma\delta - \delta S.\beta\gamma\alpha,$$

which, by formula 13, reduces to

$$2\alpha S.\beta\gamma\delta.$$

12. If, for brevity, we denote $S.\alpha\beta\gamma$, $V.\alpha\beta\gamma$ respectively by S and V, we have, by formula 7,

$$\begin{aligned} &2\alpha^2\beta^2\gamma^2 + \alpha^2(\beta\gamma)^2 + \beta^2(\alpha\gamma)^2 + \gamma^2(\alpha\beta)^2 - (\alpha\beta\gamma)^2 \\ &= 2\alpha\beta\gamma.\gamma\beta\alpha + \beta\gamma\alpha.\alpha\beta\gamma + \alpha\gamma\beta.\beta\alpha\gamma + \alpha\beta\gamma.\gamma\alpha\beta - (\alpha\beta\gamma)^2 \\ &= 2(S+V)(-S+V) + (S-V+2\alpha S\beta\gamma)(S+V) \\ &+ (-S-V+2\alpha S\beta\gamma)(-S-V+2\gamma S\alpha\beta) \\ &+ (S+V)(S-V+2\gamma S\alpha\beta) - (S+V)^2 \\ &= 4\alpha\gamma S\alpha\beta S\beta\gamma. \end{aligned}$$

The student is recommended to verify a few examples such as the above, by putting

$$\alpha = i, \; \beta = ai + bj + ck, \; \gamma = a'i + b'j + c'k,$$

with the conditions

$$a^2+b^2+c^2=1,\ a'^2+b'^2+c'^2=1.$$

The quaternion equality will then reduce itself to four algebraic equalities, one of which is obvious, and the others are

$$p^2+r^2-a'^2-a^2+2aa'm=0,$$
$$pq-mr+a'c'+ac-2ac'm=0,$$
$$qr+mp+a'b'+ab-2ab'm=0,$$

where $$m=aa'+bb'+cc',\ p=ab'-a'b',$$
$$q=bc'-b'c,\ r=ca'-c'a.$$

Ex. 13.

$$S.(\alpha-\delta)(\beta-\delta)(\gamma-\delta)=S.\alpha\beta\gamma-S.\beta\gamma\delta+S.\gamma\delta\alpha-S.\delta\alpha\beta.$$

Ex. 14. By 34. 8, we have

$$-\frac{a}{d}=\frac{S.\delta\beta\gamma}{S.\alpha\beta\gamma}=\pm\frac{BCD}{ABC};$$

therefore the same Article gives

$$\pm\alpha.BCD\pm\beta.CDA\pm\gamma.DAB\pm\delta.ABC=0;$$

and since the scalar of the product of this vector by the vector perpendicular to the plane in which A, B, C, D lie gives the right-hand side of Ex. 13, we obtain

$$\alpha.BCD-\beta.CDA+\gamma.DAB-\delta.ABC=0.$$

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